# Harmonic Functions on Manifolds of non-positive Curvature

by

Lei Ka Keung

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# 摘要

假設M為一 Cartan-Hadamard流形，即一簡單連通、完備、非緊致以及負曲率的Riemann流形。我們會總覽在這一類流形上，一些正調和函數的重要結果。第一部分將會說明在[Che]之中，在某一種Cartan-Hadamard流形上，Cheng如何解決了Dirichlet問題。第二部分假設$-b^2<K_M<-a^2$，我們將會討論幾何邊界與Martin邊界的關係以及Anderson與Schoen在這類流形上一些正調和函數的結果。最後我們會討論Freire[Fre]研究關於拋物形Martin邊界以及它對正調和函數在積流形上的應用。

# Abstract

Let $M$ be a Cartan-Hadamard manifold, i.e., simply connected, complete non-compact with non-positive curvature. We will survey some results on positive harmonic functions on this type of manifold. The first part is an exposition of the result of Cheng in [Che], which solved the Dirichlet problem at infinity for a large class of Cartan-Hadamard manifolds. The second part of this paper discuss the result of Anderson and Schoen ([A-S]) on positive harmonic functions on manifolds with curvature pinched by two negative constants and the relation between the geometric boundary and the Martin boundary. The last part of this paper will discuss the result of Freire ([Fre]) on parabolic Martin boundary and its application to positive harmonic functions on product of manifolds.

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# Contents

# Chapter 0

# Introduction

Let $M$ be a Cartan-Hadamard manifold, i.e., a simply connected, complete, non-compact Riemannian manifold with non-positive sectional curvature. In this paper, we will survey some important results on positive harmonic functions on this type of manifolds. It is well known that given such a manifold $M$, we can compactify it using equivalent classes of geodesic rays. The set of all equivalent classes are called the geometric boundary of $M$ and is denoted by $M(\infty)$ (see chapter one for details). Write $\overline{M} = M \cup M(\infty)$. For example, let $M$ be the two dimensional hyperbolic plane $\mathbb{H}^2$. Then, $\overline{\mathbb{H}^2}$ can be identified with the closed unit disc, and $\mathbb{H}^2(\infty)$ can be identified with the unit circle $\mathbb{S}^1$. If $M = \mathbb{H}^2$, it is classical that given a continuous function $f$ on $\mathbb{S}^1$, we can find a solution to the equation

$$\begin{cases} \Delta u = 0, & \text{on } M, \\ u = f, & \text{on } M(\infty). \end{cases} \tag{0.1}$$

Hence, it is natural to ask if this is true for a general Cartan-Hadamard manifold $M$. This Dirichlet problem of harmonic functions with prescribed boundary data at infinity is a very interesting problem in geometry and analysis. This problem have been solved by different people under different curvature assumptions:

In [Sul1], Sullivan uses probabilistic approach, solved the problem on a Cartan-

Hadamard manifold with sectional curvature bounded by two negative constants.

In [And], Anderson solved the problem independently under the same assumptions by a more geometric approach in the same line as in [Cho] for rotational symmetric metric.

In [A-S], under the same assumption, Anderson and Schoen solved the problem by estimating the Hessian and gradient of the radial extension of the boundary data. The curvature assumption in these results are reasonable since in the case that $M = \mathbb{R}^n$, the Dirichlet problem is not solvable. In fact, all bounded positive harmonic functions on $\mathbb{R}^n$ are constants.

In [S-Y], Schoen and Yau proved that if $(M, d\tilde{s}^2)$ complete, simply connected with sectional curvatures bounded by two negative constants, $ds^2$ is another metric on $M$ uniformly equivalent to $d\tilde{s}^2$, i.e., there exists constant $A > 0$ such that $\frac{1}{A}ds^2 \leq d\tilde{s}^2 \leq Ads^2$, and if the sectional curvatures of $(M, ds^2)$ are bounded from above and below, the injectivity radius of $(M, ds^2)$ is bounded away from zero, then, (0.1) is solvable, where $\Delta$ is the Laplacian operator on $(M, ds^2)$.

In this result, the curvature assumption is relaxed but there is still a restriction on injectivity radius. In [Anc], Ancona showed that the restriction on injectivity radius in fact can be removed.

The first part of this paper is to give an exposition on a more general result of Cheng [Che]. Cheng showed that the Dirichlet problem (0.1) can be solved under the following assumptions: $\lambda_1(M) > 0$, there exist $x_0 \in M$, constant $C \geq 1$ such that $\forall x \in M$, we have $|K(\sigma)| \leq C|K(\sigma')|$, where $\sigma$, $\sigma'$ are plane sections at $x$ containing the tangent vector of the geodesic joining $x_0$ to $x$. Note that in case the sectional curvature is bounded by two negative constants, the above conditions are automatically satisfied. Furthermore, the above assumptions can be relaxed by requiring only a uniformly equivalent metric satifies the above (Corollary 1.1). So, the method used in this paper in fact gives a generalization of the results on [Anc], [Sul1].

In the second part of this paper, we will discuss positive harmonic functions on Cartan-Hadamard manifolds with sectional curvature bounded by two negative constants. R.S. Martin ([Mar]) introduced the idea of Martin boundary and gave a representation formula for positive harmonic functions on a domain $\Omega \subset \mathbb{R}^3$ with positive Green's functions in the early 40's. He showed that if $u$ is a positive harmonic function on $M$, then $\exists$ a positive Borel measure $\mu$ on the Martin boundary $\widetilde{M}$ such that

$$u(x) = \int_{\widetilde{M}} K(x,y) d\mu(y),$$

where $K(x,y)$ is a kernel function, $x \in M$, $y \in \widetilde{M}$.

However, such $\mu$ is not unique unless we restrict to a subset of $\widetilde{M}$ which corresponds to the set of the so called minimal positive harmonic functions. The Martin boundary is defined in terms of the limiting behaviour of the normalized Green's function. In the case of the unit disc in $\mathbb{R}^2$, the representation formula is just the Poisson formula. The idea of Martin can be generalized to a Cartan-Hadamard manifold $M$. One then asked whether such a formula is true for such an $M$ with $-a^2 \leq K_M \leq -b^2 < 0$. In [A-S], Anderson and Schoen showed that on such a manifold, there is a homeomorphism between the Martin boundary $\widetilde{M}$ and the geometric boundary $M(\infty)$. In fact, one can show that $M(\infty)$ possesses a $C^\alpha$ structure, where $0 < \alpha < 1$, depending only on the dimension of $M$ and the sectional curvature bounds. Then, since $\widetilde{M}$ is homeomorphic to $M(\infty)$, $\widetilde{M}$ also possesses a $C^\alpha$ structure and we also have a representation formula for positive harmonic functions on the geometric boundary. Moreover, each point on $M(\infty)$ corresponds to a minimal positive harmonic function.

The third part of this paper is a discussion on generalizing the idea of Martin boundary to the parabolic Martin boundary. Using this, Freire ([Fre]) studies positive harmonic functions on a product of manifolds $M = M_1 \times M_2$, where $M_1$ and $M_2$ are complete, non-compact manifolds with Ricci curvature bounded below. He proved that:

(i) Each minimal positive harmonic function $f$ on $M$ splits as a product

$$f(x) = K^{\lambda_1}(x^1)K^{\lambda_2}(x^2),$$

where $\lambda_i \geq \lambda_0(M_i)$, $K^{\lambda_i} \in \widetilde{M}_1^{\lambda_i}(M_i)$ for $i = 1, 2$, and $\lambda_1 + \lambda_2 = 0$.

(ii) Conversely, each product as above is a minimal positive harmonic function on $M$.

In case $f$ is bounded, if $\Delta_1$ and $\Delta_2$ are the Laplacian operators on $M_1$ and $M_2$ respectively, we have $\Delta_1 f = \Delta_2 f = 0$.

# Chapter 1

# Dirichlet Problem at infinity

## 1.1 The Geometric Boundary

In this chapter, we will discuss how we can solve the Dirichlet problem for harmonic functions at infinity on a complete non-compact manifold $M$ with certain curvature . Let us first define the geometric boundary of a complete non-compact manifold.

DEFINITION **1.1.** *Let $M$ be a complete non-compact Riemannian manifold. The geometric boundary is defined to be the set of all equivalent classes of geodesic rays on $M$, where two geodesics $\gamma_1$, $\gamma_2 : [0, \infty) \to M$ are equivalent iff $|\gamma_1(t) - \gamma_2(t)| \leq C$ for some constant $C$ independent of $t$. The geometric boundary of $M$ is denoted by $M(\infty)$.*

$M \cup M(\infty)$, together with the cone topology, gives a compactification of $M$: Let $C_{x_0}(v, \theta)$ be a cone at $x_0 \in M$ with direction $v \in T_{x_0}M$ and angle $\theta$, i.e., $C_{x_0}(v, \theta) = \{x \in M : \angle_{x_0}(v, T_{x_0x}) < \theta$, where $T_{x_0x}$ is the tangent vector of the unique minimizing geodesic joining $x_0$ and $x\}$. Let $T_{x_0}(v, \theta, R) = C_{x_0}(v, \theta) \backslash B(x_0, R)$ denote a truncated cone. $T_{x_0}(v, \theta, R)$ together with the geodesic balls $B_x(r)$, $r > 0$, $x \in M$ form a local basis for the cone topology. Let $\zeta : [0, 1] \to [0, \infty]$ be a fixed homeomorphism. The map $E_\zeta : B_1 \subset T_{x_0}M \to M$

given by $E_\zeta(v) = exp_{x_0}(\zeta(|v|)v)$ is a homeomorphism of the open unit ball $B_1$ in $T_{x_0}M$ onto $M$. Further, $E_\zeta$ extends to a homeomorphism of the sphere $S_1 = \partial B_1$ onto $M(\infty)$. We can see that the cone topology is well defined, i.e., independent of the bas point $x_0$ (see [E-O]) so that $M(\infty)$ gives a topological compactification of $M$.

Consider on an $n$-dimensional complete, simply connected Riemannian manifold $M$ with non-positive sectional curvature, the geometric boundary can be identified with a sphere, we call it the sphere at infinity (if we fix an origin in $M$, then, in each equivalent class of geodesic rays, there is one and only one representative starting at that point). In case of $-b^2 \leq K_M \leq -a^2 < 0$, $M(\infty)$ itself admits a $C^\alpha$ structure.

PROPOSITION **1.1.** *Let $M^n$ be a complete, simply connected negatively curved manifold with $-\infty \leq -b^2 \leq K_M \leq -a^2 < 0$. Then the geometric boundary $M(\infty)$ is a well-defined $C^\alpha$-structure, $\alpha = \frac{a}{b}$.*

We need the following Lemmas first:

LEMMA **1.1.** *Let $\Delta ABC$ be a geodesic triangle in $M$ with $\rho(A,B) = r$, $\rho(A,C) = \sigma$, $\rho(B,C) = s$ and $\angle A = \theta$, then,*

$$\begin{aligned} \cos\theta &\geq \coth ar \coth a\sigma - \frac{\cosh as}{\sinh ar \sinh a\sigma} \\ \cos\theta &\leq \coth br \coth b\sigma - \frac{\cosh bs}{\sinh br \sinh b\sigma} \end{aligned}$$

*where $\rho$ is the distance function on $M$.*

*Proof of Lemma 1.1.* Consider on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ with the metric

$$ds^2 = \frac{4}{a^2(1-|r|^2)^2}(dx^2 + dy^2)$$

evaluating at $r \in D$.

Let $\gamma(t) = (tx_0, ty_0)$. Then, $\gamma'(t) = x_0\frac{\partial}{\partial x_1} + y_0\frac{\partial}{\partial x_2}$. Hence,

$$|\gamma'(t)|^2 = x_0^2 g_{11} + y_0^2 g_{22} = (x_0^2 + y_0^2)\frac{4}{a^2(1-t^2(x_0^2+y_0^2))^2}, 0 \leq t \leq 1.$$

$$\begin{aligned}
\int_0^1 |\gamma'(t)|dt &= \int_0^1 \frac{2|r|}{a(1-t^2|r|^2)}dt \\
&= \frac{1}{a}\log\frac{1+|r|}{1-|r|}
\end{aligned}$$

Therefore, define the distance function $\rho(0,r)$ to be $\frac{1}{a}\log\frac{1+|r|}{1-|r|}$

Now, WLOG, suppose $r_1$ is real, $r_2 = |r_2|e^{i\theta}$. Let $\sigma = \rho(0,r_1) = \frac{1}{a}\log\frac{1+r_1}{1-r_1}, \delta = \rho(0,r_2) = \frac{1}{a}\log\frac{1+|r_2|}{1-|r_2|}$. Then, $r_1 = \tanh\frac{a\sigma}{2}, |r_2| = \tanh\frac{a\delta}{2}$. Consider the isometry on D,

$$f(z) = \frac{z-r_1}{1-r_1 z},$$

let $s = \rho(r_1, r_2)$. Then

$$\begin{aligned}
s = \rho(r_1,r_2) &= \rho(0, \frac{r_2-r_1}{1-r_1r_2}) \\
&= \frac{1}{a}\log\frac{1+|\frac{r_2-r_1}{1-r_1r_2}|}{1-|\frac{r_2-r_1}{1-r_1r_2}|}.
\end{aligned}$$

Therefore, $|\frac{r_2-r_1}{1-r_1r_2}| = \tanh\frac{as}{2}$,

$$\begin{aligned}
\tanh^2\frac{as}{2} &= \frac{||r_2|e^{i\theta}-r_1|^2}{|1-r_1|r_2|e^{i\theta}|^2} \\
&= \frac{||r_2|\cos\theta - r_1 + i|r_2|\sin\theta|^2}{|1-r_1|r_2|\cos\theta - ir_1|r_2|\sin\theta|^2} \\
&= \frac{(|r_2|\cos\theta - r_1)^2 + (|r_2|\sin\theta)^2}{(1-r_1|r_2|\cos\theta)^2 + (r_1|r_2|\sin\theta)^2} \\
&= \frac{|r_2|^2 - 2r_1|r_2|\cos\theta + r_1^2}{1-2r_1|r_2|\cos\theta + r_1^2|r_2|^2} \\
\frac{1}{\cosh as} &= \frac{1-\tanh^2\frac{as}{2}}{1+\tanh^2\frac{as}{2}} \\
&= \frac{(1-r_1^2)(1-|r_2|^2)}{(1+r_1^2)(1+|r_2|^2) - 4r_1|r_2|\cos\theta} \\
&= \frac{(1-\tanh^2\frac{a\sigma}{2})(1-\tanh^2\frac{a\delta}{2})}{(1+\tanh^2\frac{a\sigma}{2})(1+\tanh^2\frac{a\delta}{2}) - 4\tanh\frac{a\sigma}{2}\tanh\frac{a\delta}{2}\cos\theta}.
\end{aligned}$$

Hence,

$$\begin{aligned}\cosh as &= (\cosh^2\frac{a\sigma}{2}+\sinh^2\frac{a\sigma}{2})(\cosh^2\frac{a\delta}{2}+\sinh^2\frac{a\delta}{2}) - \\ &\quad 4\sinh\frac{a\sigma}{2}\sinh\frac{a\delta}{2}\cosh\frac{a\sigma}{2}\cosh\frac{a\delta}{2}\cos\theta \\ &= \cosh a\sigma\cosh a\delta - \sinh a\sigma\sinh a\delta\cos\theta\end{aligned}$$

Therefore,

$$\cos\theta = \coth a\sigma\coth a\delta - \frac{\cosh as}{\sinh a\sigma\sinh a\delta}$$

The lemma then follows by applying Toponogov's Comparison Theorem. □

LEMMA **1.2.** *Let $O$, $x_1$, $x_2$ be 3 points in $M$ with $\rho(O,x_1)=\rho(O,x_2)=r$, $\gamma_i$ be the geodesic ray from $O$ to $x_i$, $i=1,2$, $\theta$ be the angle between $\gamma_i$ at $O$. Then for $r$ large enough and $\theta$ small enough, we have,*

$$2r+\frac{2}{a}(\log\theta-1)\le\rho(x_1,x_2)\le 2r+\frac{2}{b}(\log\theta+1)$$

*Proof of Lemma 1.2.* Putting $\sigma=r$ and $\rho(x_1,x_2)=s$ in the first part of Lemma 1.1, we have

$$\begin{aligned}\cos\theta\sinh^2 ar &\ge \cosh^2 ar-\cosh as \\ &= 1+\sinh^2 ar-\cosh as.\end{aligned}$$

Therefore, $\cosh as - 1 \ge \sinh^2 ar(1-\cos\theta)$

$$l.h.s. = \frac{e^{as}+e^{-as}}{2}-1=\frac{e^{as}-2+e^{-as}}{2}=\frac{\left(e^{\frac{as}{2}}-e^{\frac{-as}{2}}\right)^2}{2}=2\sinh^2\frac{as}{2}.$$

Hence, $4\sinh^2\frac{as}{2}\ge\sinh^2 ar\cdot 2(1-\cos\theta)$.

Next, consider

$$\begin{aligned}e^{ar}-e^{-ar} &= 2\cdot\frac{e}{2}e^{ar-1}-e^{-ar} \\ &\ge 2C_1e^{ar-1}\end{aligned}$$

for some $1 < C_1 < \frac{e}{2}$, $r$ large enough. Hence, $\frac{\sinh ar}{C_1} \geq e^{ar-1}$. Then,

$$\begin{aligned} 2C_1^2(1-\cos\theta) &= 2C_1^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right) \\ &= C_1^2\theta^2 + 2C_1^2\left(-\frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right) \geq \theta^2 \end{aligned}$$

for $\theta$ small enough. Therefore, we choose $1 < C < \frac{e}{2}$. We have, for $r$ large enough, $\theta$ small enough, $\sinh^2 ar \cdot 2(1-\cos\theta) = \dfrac{\sinh^2 ar}{C^2} \cdot 2C^2(1-\cos\theta) \geq e^{2(ar-1)}\theta^2$. Hence, $4\sinh^2 \dfrac{as}{2} \geq e^{2(ar-1)}\theta^2$.

$$\begin{aligned} 2\log(e^{\frac{as}{2}} - e^{-\frac{as}{2}}) &\geq 2(ar-1) + 2\log\theta = 2ar + 2(\log\theta - 1) \\ 2\log(e^{\frac{as}{2}} - e^{-\frac{as}{2}}) &= 2\log\left(e^{\frac{as}{2}}(1-e^{-as})\right) = as + 2\log(1-e^{-as}) \leq as \end{aligned}$$

Therefore, we get

$$s \geq 2r + \frac{2}{a}(\log\theta - 1)$$

Using similar arguments as above, we have

$$\begin{aligned} 4\sinh^2\frac{bs}{2} &\leq \sinh^2 br \cdot 2(1-\cos\theta) \\ &= \left(\frac{1}{2}(e^{br} - e^{-br})\right)^2 \cdot \left(\theta^2 + 2\left(-\frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right)\right) \\ &\leq \frac{1}{4}e^{2br}\theta^2 \\ 2\log\left(e^{\frac{bs}{2}} - e^{\frac{-bs}{2}}\right) &\leq 2br + 2(\log\theta) - \log 4 \\ bs + 2\log\left(1 - e^{bs}\right) &\leq 2br + 2(\log\theta + 1) - 2 - \log 4 \\ s &\leq 2r + \frac{2}{b}\left(\log\theta + 1\right) + \frac{1}{b}\left(-2 - \log 4 - 2\log\left(1 - e^{-bs}\right)\right) \end{aligned}$$

Since as $r \to \infty$, $s = \rho(x_1, x_2) \to \infty$, $\log(1-e^{-bs}) \to 0$ is small comparing with $r$, we have

$$s \leq 2r + \frac{2}{b}\left(\log\theta + 1\right),$$

for $s$ large enough. $\square$

*Proof of Proposition 1.1.* Let $x_1, x_2 \in M$, $v, w \in T_{x_1}M$, $|v| = |w| = 1$, $\theta$ denote the angle between $v$ and $w$ at 0. Let $\gamma_v(t)$, $\gamma_w(t)$ be geodesic rays at $x_1$ with tangent vectors $v$, $w$ respectively. Then, by Lemma 1.2, for $\bar{t}$ large enough,

$$\rho(\bar{t}) = \rho\left(\gamma_v(\bar{t}), \gamma_w(\bar{t})\right) \leq 2\bar{t} + \frac{2}{b}(\log\theta + 1)$$

Let $\tilde{\gamma}_v(t)$, $\tilde{\gamma}_w(t)$ be geodesic rays at $x_2$ such that $\tilde{\gamma}_v(t_1) = \gamma_v(\bar{t})$, $\tilde{\gamma}_w(t_2) = \gamma_w(\bar{t})$. WLOG, suppose $t_2 \leq t_1$.

$$\begin{aligned}
\tilde{\rho}(t_2) &\stackrel{\text{def}}{=} \rho(\tilde{\gamma}_v(t_2), \tilde{\gamma}_w(t_2)) \\
&\geq 2t_2 + \frac{2}{a}\log\tilde{\theta}_t - \frac{2}{a} \\
\frac{2}{a}\log\tilde{\theta}_{\bar{t}} &\leq \tilde{\rho}(t_2) - 2t_2 + \frac{2}{a}
\end{aligned}$$

Where $\tilde{\theta}_{\bar{t}}$ is the angel between $\tilde{\gamma}_v(t_1)$ and $\tilde{\gamma}_w(t_2)$ at $x_2$. Note that using triangle inequalities, we have,

$$\begin{aligned}
\bar{t} \leq t_2 + \rho(x_1, x_2) \quad &, \quad t_1 \leq \bar{t} + \rho(x_1, x_2) \\
\therefore -t_2 \leq -\bar{t} + \rho(x_1, x_2) \quad &, \quad t_1 - t_2 \leq 2\rho(x_1, x_2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{2}{a}\log\tilde{\theta}_{\bar{t}} &\leq \rho(\bar{t}) + (t_1 - t_2) - 2\bar{t} + 2\rho(x_1, x_2) + \frac{2}{a} \\
&\leq \rho(\bar{t}) - 2\bar{t} + 4\rho(x_1, x_2) + \frac{2}{a} \\
&\leq 2\bar{t} + \frac{2}{b}\log\theta - 2\bar{t} + \left(\frac{2}{b} + \frac{2}{a} + 4\rho(x_1, x_2)\right) \\
\text{Hence,} \qquad \log\tilde{\theta}_{\bar{t}} &\leq \frac{a}{b}\log\theta + \tilde{C}(a, b, \rho(x_1, x_2)) \\
\tilde{\theta}_{\bar{t}} &\leq C_1\theta^{\frac{a}{b}}
\end{aligned}$$

□

## 1.2 Dirichlet Problem

The Dirichlet problem of harmonic functions with prescribed boundary data at the geometric boundary is interesting in geometry and analysis, it was solved in various dimensions with different curvature requirements.

In [Sul1], Sullivan uses probabilistic approach, solved the problem on a simply connected manifold with sectional curvature bounded by two negative constants.

In [And], Anderson solved the problem independently under the same assumptions by a more geometric approach in the same line as in [Cho] for rotational symmetric metric.

In [A-S], the problem under a requirement of pinching by two negative constants was solved by estimating the Hessian and gradient of the radial extension of the boundary data.

In this section, we will follow the method of Cheng, which solved this problem in a more general setting [Che]. We will first give an $L^p$ estimation:

Proposition **1.2.** *Let $M$ be a complete non-compact manifold. Assume that $\lambda_1(M) > 0$. Suppose $\Omega$ is a compact sub domain of $M$ such that $\partial\Omega$ smooth. Let $f \in C^\infty(M)$ and $u$ satisfies*

$$\begin{cases} \Delta u = -\Delta f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{1.1}$$

*Then, for $p \geq 2$,*

$$\int_\Omega |u|^p \leq c \int_\Omega |\nabla f|^p,$$

*where $c$ is a positive constant depending only on $p$ and $\lambda_1(M)$.*

*Proof.* Multiplying $(\text{sgn}u)|u|^{p-1}$ to 1.1, we get

$$\begin{aligned} (\text{sgn}u)|u|^{p-1}\nabla u &= -(\text{sgn}u)|u|^{p-1}\nabla f \\ \int_\Omega |u|^{p-1}\nabla|u| &= -(\text{sgn}u)\int_\Omega |u|^{p-1}\nabla f \end{aligned}$$

Therefore,

$(|u|^{p-1}\nabla|u|)|_{\partial\Omega} - \int_\Omega \nabla|u|\,((p-1)|u|^{p-2}\nabla|u|)$

$= (-(\text{sgn}u)\left[|u|^{p-1}\nabla f)|_{\partial\Omega} - \int_\Omega \nabla f\,((p-1)|u|^{p-2}\nabla|u|)\right]$

Hence,

$$\begin{aligned}\int_\Omega |u|^{p-2}|\nabla|u||^2 &= -(\text{sgn}u)\int_\Omega |u|^{p-2} < \nabla|u|, \nabla f > \\ &\leq \int_\Omega |u|^{p-2}|\nabla|u|||\nabla f| \\ &\leq \frac{1}{2}\int_\Omega |u|^{p-2}(|\nabla|u||^2 + |\nabla f|^2)\end{aligned}$$

Finally, we have

$$\int_\Omega |u|^{p-2}|\nabla|u||^2 \leq \int_\Omega |u|^{p-2}|\nabla f|^2$$

Consider l.h.s. of the inequality,

$$\frac{4}{p^2}\int_\Omega |\nabla|u|^{\frac{p}{2}}|^2 = \frac{4}{p^2}\int_\Omega \frac{p^2}{4}(|u|^{\frac{p}{2}-1})^2|\nabla|u||^2 = \int_\Omega |u|^{p-2}|\nabla|u||^2 \leq \int_\Omega |u|^{p-2}|\nabla|u||^2$$

r.h.s.:

$$\int_\Omega (|u|^p)^{\frac{p-2}{p}}(|\nabla f|^p)^{\frac{2}{p}} \leq \left(\int_\Omega |u|^p\right)^{\frac{p-2}{p}}\left(\int_\Omega |\nabla f|^p\right)^{\frac{2}{p}}$$

Therefore,

$$\begin{aligned}\frac{4}{p^2}\lambda_1(M)\int_\Omega |u|^p &\leq \frac{4}{p^2}\frac{\int_\Omega |\nabla|u|^{\frac{p}{2}}|^2}{\int_\Omega ||u|^{\frac{p}{2}}|^2}\int_\Omega |u|^p \\ &= \frac{4}{p^2}\int_\Omega |\nabla|u|^{\frac{p}{2}}|^2 \\ &\leq \int_\Omega |u|^{p-2}|\nabla|u||^2 \\ &\leq \int_\Omega |u|^{p-2}|\nabla f|^2 \\ &\leq \left(\int_\Omega |u|^p\right)^{\frac{p-2}{p}}\left(\int_\Omega |\nabla f|^p\right)^{\frac{2}{p}}\end{aligned}$$

Thus,

$$\int_\Omega |u|^p \leq \left(\frac{p^2}{4\lambda_1(M)}\right)^{\frac{p}{2}}\int_\Omega |\nabla f|^p$$

$\square$

THEOREM **1.1.** *Assume* $\lambda_1(M) > 0$. *Let* $f \in C^\infty(M), \int_M |\nabla f|^p < +\infty$ *for* $p \geq 2$, $u_R$ *satisfies* $\Delta u_R = -\Delta f$ *on* $B(x_0, R)$ *and* $u_R = 0$ *on* $\partial B(x_0, R)$ *for* $x_0 \in M$, $R > 0$. *Then,* $\exists$ *a subsequence of* $\{u_R\}$ *that converges uniformly on compact subsets to a smooth function* $u$ *on* $M$ *such that* $\Delta u = -\Delta f$ *on* $M$ *and* $\int_M |u|^p < +\infty$

We'll first prove the following:

CLAIM **1.1.** $\forall \Omega \subset\subset M$, $\sup_\Omega |u_R + f| \leq C(\Omega)$, $\Omega \subset\subset M$, *where* $C(\Omega)$ *is a constant depends on* $\Omega$ *only.*

*Proof.* Let $\Omega \subset\subset M$, $\mathrm{Ric}(\Omega) \geq -K$. Since $u$ is non-negative and sub-harmonic on $M$, for $\tau \in (0, \frac{1}{2})$, we have, by sub-mean value inequality ([S-Y] Page 77),

$$\begin{aligned} \sup_{B((1-\tau)R)} |u_R|^2 &\leq C_1 \tau^{-C_2(1+\sqrt{K}R)} \frac{1}{|B(R)|} \int_{B(R)} |u_R|^2 dV \\ &= C(\tau, K, R, B(R)) \int_{B(R)} |u_R|^2 dV \end{aligned}$$

Since $\Omega$ is compact, we can choose a finite cover of $\Omega$ by balls. Hence, we can choose a constant $C(\Omega)$ depending on $\Omega$ only such that

$$\sup_\Omega |u_R|^2 \leq C(\Omega)$$

$R$ large enough. □

*Proof of Theorem.* Now, we have $\Delta(u_R + f) = 0$, $u_R = 0$ on $\partial B(x_0, R)$ for $x_0 \in M$ fixed and $\sup_{B(x_0,R)} |u|$ bound by a constant depending only on $R$. Using Schauder interior estimates ([G-T] Page 89), first, second and third derivatives of $u_R + f$ are also bounded on compact sets. Then, the Arzela Ascoli's Theorem implies that there exists a limit, say, $u + f$ of $\{u_R + f\}_{R\to\infty}$ such that $\Delta(u + f) = 0$. Since $f \in C^\infty(M)$ and $\int_M |\nabla f|^p < +\infty$, prop 1.2 implies

$$\begin{aligned} \int_M |u|^p &\leq C(p, \lambda_1(M)) \int_M |\nabla f|^p \\ &\leq C_1. \end{aligned}$$

where $C_1$ is a constant. Therefore, $\int_M |u|^p$ bounded on $M$. □

We will now show under certain assumptions that $u+f$ obtained from the previous section assumes the boundary data. Using the property $\int_M |u|^p < +\infty$, we will show that $u(x) \to 0$ as $x \to \infty$.

THEOREM **1.2.** *Suppose $\exists S_1 > 0$ such that $\forall \phi \in C_0^\infty(M)$, $S_1 \left(\int_M |\phi|^\kappa\right)^{\frac{1}{\kappa}} \leq \int_M |\nabla \phi|$, $\kappa = \frac{n}{n-1}$, $u$ bounded in $L^p(M)$ with $\Delta u = -\Delta f$ such that $|\nabla f| \in L^p$ is bounded for $p \geq 2$. Then, $\exists$ constant $\delta \in (0,1)$ such that for $z \in M$, $\tau_0 \in (0,1)$, $R_0 > 0$, we have for $R \in (0, R_0)$,*

$$\sup_{B(z,\tau_o R)} |u|^p \leq \frac{C}{\tau_0^n R^n} \left(\int_{B(z_0,R)} |u|^p\right)^\delta,$$

*where $C$ is a constant depending on $n$, $p$, $R_0$, $\sup_M |u|$, $\sup_M |\nabla f|$, $\int_M |u|^p$, $\int_M |\nabla f|^p$.*

*Proof.* WLOG, assume that $|u| \leq 1$, $\int |u|^p \leq 1$, we have $\Delta u = -\Delta f$. Suppose $\eta$ is a cut-off function, then,

$$\int_\Omega \eta^2(\text{sgn}u)|u|^{q-1}\Delta u = \int_\Omega -\eta^2(\text{sgn}u)|u|^{q-1}\Delta f, q \geq 2.$$

$l.h.s. = \int_{\partial\Omega} \eta^2(\text{sgn}u)|u|^{q-1}\frac{\delta u}{\delta n} - \int_\Omega < \nabla(\eta^2(\text{sgn}u)|u|^{q-1}, \nabla u >$

Suppose $\Omega$ large enough such that $\eta$ is zero at $\partial\Omega$, denote $\Omega_+$ and $\Omega_-$ be subsets of $\Omega$ such that $u$ is positive and negative respectively, then,

$$\begin{aligned} l.h.s. &= -\int_{\Omega_+} < \nabla(\eta^2 u^{q-1}), \nabla u > + \int_{\Omega_-} < \nabla(\eta^2(-u)^{q-1}), \nabla u > \\ &= -\int_{\Omega_+} 2\eta u^{q-1} < \nabla\eta, \nabla u > - \int_{\Omega_+} (q-1)\eta^2 u^{q-2} < \nabla\eta, \nabla u > \\ &\quad + \int_{\Omega_-} 2\eta(-u)^{q-1} < \nabla\eta, \nabla u > + \int_{\Omega_-} (q-1)\eta^2(-u)^{q-2} < \nabla\eta, \nabla u > \\ &= -2\int_\Omega \eta(\text{sgn}u)|u|^{q-1} < \nabla\eta, \nabla u > \\ &\quad -(q-1)\int_\Omega \eta^2(\text{sgn}u)|u|^{q-2} < \nabla(\text{sgn}u), \nabla u > \end{aligned}$$

Similarly,

$$r.h.s. = 2\int_\Omega \eta(\text{sgn}u)|u|^{q-1} < \nabla\eta, \nabla f > +(q-1)\int_\Omega \eta^2(\text{sgn}u)|u|^{q-2} < \nabla f, \nabla|u| > .$$

Then,

$$\begin{aligned}
& (q-1)\int_\Omega \eta^2(\mathrm{sgn}u)|u|^{q-2} < \nabla|u|, \nabla u > \\
= \;& (q-1)\int_\Omega \eta^2|u|^{q-2}|\nabla|u||^2 \\
= \;& -(q-1)\int_\Omega \eta^2(\mathrm{sgn}u)|u|^{q-2} < \nabla f, \nabla|u| > -2\int_\Omega \eta(\mathrm{sgn}u)|u|^{q-1} < \nabla\eta, \nabla f > \\
& -2\int_\Omega \eta(\mathrm{sgn}u)|u|^{q-1} < \nabla\eta, \nabla u > \\
\leq \;& (q-1)\left[\int_\Omega \eta^2|u|^{q-2}|\nabla f|\cdot|\nabla u|\right] + 2\left[\int_\Omega \eta|u|^{q-1}|\nabla\eta|\cdot|\nabla f|\right] \\
& +2\left[\int_\Omega \eta|u|^{q-1}|\nabla\eta|\cdot|\nabla u|\right] \\
\leq \;& (q-1)\left(\int_\Omega \eta^2|u|^{q-2}|\nabla f|^2\right)^{\frac{1}{2}}\left(\int_\Omega \eta^2|u|^{q-2}|\nabla u|^2\right)^{\frac{1}{2}} + 2\left(\int_\Omega \eta^2|u|^{q-2}|\nabla f|^2\right)^{\frac{1}{2}} \\
& \times\left(\int_\Omega |u|^q|\nabla\eta|^2\right)^{\frac{1}{2}} + 2\left(\int_\Omega \eta^2|u|^{q-2}|\nabla u|^2\right)^{\frac{1}{2}}\left(\int_\Omega |u|^q|\nabla\eta|^2\right)^{\frac{1}{2}} \\
\leq \;& (q-1)\left(\int_\Omega \eta^2|u|^{q-2}|\nabla f|^2\right) + \frac{q-1}{4}\left(\int_\Omega \eta^2|u|^{q-2}|\nabla u|^2\right) \\
& +(q-1)\left(\int_\Omega \eta^2|u|^{q-2}|\nabla f|^2\right) + \frac{1}{q-1}\left(\int_\Omega |u|^q|\nabla\eta|^2\right) \\
& +\frac{q-1}{4}\left(\int_\Omega \eta^2|u|^{q-2}|\nabla u|^2\right) + \frac{4}{q-1}\left(\int_\Omega |u|^q\nabla\eta|^2\right)
\end{aligned}$$

$$\therefore \frac{q-1}{2}\int_\Omega \eta^2|u|^{q-2}|\nabla u|^2 \leq 2(q-1)\int_\Omega \eta^2|u|^{q-2}|\nabla f|^2 + \frac{5}{q-1}\int_\Omega |u|^q|\nabla\eta|^2, q \geq 2$$

Note that $|\nabla|u|^{\frac{q}{2}}|^2 = \left(\frac{q}{2}|u|^{\frac{q}{2}-1}\nabla|u|\right)^2 = \frac{q^2}{4}|u|^{q-2}|\nabla|u||^2 \leq (q-1)^2|u|^{q-2}|\nabla|u||^2$, then,

$$\begin{aligned}
\int \eta^2|\nabla|u|^{\frac{q}{2}}|^2 &\leq (q-1)^2\int \eta^2|u|^{q-2}|\nabla|u||^2 \\
&= 2(q-1)\left[\frac{q-1}{2}\int \eta^2|u|^{q-2}|\nabla|u||^2\right] \\
&\leq 4(q-1)^2\int \eta^2|u|^{\frac{q}{2}}|\nabla f|^2 + 10\int |u|^q|\nabla\eta|^2
\end{aligned}$$

Consider for some $\varepsilon > 0$,

$$\begin{aligned}
|\nabla(\eta^2|u|^q)| &\leq 2\eta|u|^q|\nabla\eta| + q\eta^2|u|^{q-1}|\nabla|u|| \\
&= 2\eta|u|^q|\nabla\eta| + 2\eta^2|u|^{\frac{q}{2}}\left(\frac{q}{2}|u|^{\frac{q}{2}-1}|\nabla|u||\right) \\
&= 2|u|^{\frac{q}{2}}|\nabla\eta|\cdot\eta|u|^{\frac{q}{2}} + 2\eta|\nabla|u|^{\frac{q}{2}}|\cdot\eta|u|^{\frac{q}{2}} \\
&\leq \varepsilon|u|^q|\nabla\eta|^2 + \frac{1}{\varepsilon}\eta^2|u|^q + \varepsilon\eta^2|\nabla|u|^{\frac{q}{2}}|^2 + \frac{1}{\varepsilon}\eta^2|u|^q \\
&= \varepsilon\eta^2|\nabla|u|^{\frac{q}{2}}|^2 + \frac{2}{\varepsilon}\eta^2|u|^q + \varepsilon|u|^q|\nabla\eta|^2.
\end{aligned}$$

By hypothesis, we have,

$$\begin{aligned}
& S_1\left(\int_{B(z,\theta R)}(|u|^q)^\kappa\right)^{\frac{1}{\kappa}} \\
\leq\ & S_1\left(\int_M(\eta^2|u|^q)^\kappa\right)^{\frac{1}{\kappa}} \\
\leq\ & \int_M \nabla(\eta^2|u|^q) \\
\leq\ & \int_M\left[\varepsilon\eta^2|\nabla|u|^{\frac{q}{2}}|^2 + \frac{2}{\varepsilon}\eta^2|u|^q + \varepsilon|u|^q|\nabla\eta|^2\right] \\
\leq\ & 4\varepsilon(q-1)^2\int_M\eta^2|u|^{q-2}|\nabla f|^2 + 10\varepsilon\int_M|u|^q|\nabla\eta|^2 + \frac{2}{\varepsilon}\int_M\eta^2|u|^q + \varepsilon\int_M|u|^q|\nabla\eta|^2 \\
=\ & 4\tau R(q-1)^2\int_{B(z,(\theta+\tau)R)}|u|^{q-2}|\nabla f|^2 + \frac{2}{\tau R}\int_{B(z,(\theta+\tau)R)}|u|^q + 11\tau R\int_M|u|^q|\nabla\eta|^2 \\
\leq\ & 4\tau R(q-1)^2\int_{B(z,(\theta+\tau)R)}|u|^{q-2}|\nabla f|^2 + \frac{13}{\tau R}\int_{B(z,(\theta+\tau)R)}|u|^q,
\end{aligned}$$

taking $\eta(x) = \begin{cases} 1 & \text{for} \quad d(x,z) \leq \theta R \\ 0 & \text{for} \quad d(x,z) \geq (\theta+\tau)R \\ \frac{(\theta+\tau)R - d(x,z)}{\tau R} & \text{otherwise} \end{cases}$

Now, consider $\dfrac{|\nabla f}{\sup|\nabla f|} \leq 1$, then, for $p \leq q$, $\displaystyle\int\left(\frac{|\nabla f}{\sup|\nabla f|}\right)^q \leq \int\left(\frac{|\nabla f}{\sup|\nabla f|}\right)^p$
Hence, $\int|\nabla f|^q \leq (\sup|\nabla f|)^{q-p}\int|\nabla f|^p$, which implies
$\left(\int|\nabla f|^q\right)^{\frac{2}{q}} \leq (\sup|\nabla f|)^{2(1-\frac{p}{q})}\|\nabla f\|_p^{2-\frac{p}{q}} = K$ constant depending on $\sup|\nabla f|$, $\|\nabla f\|_p$.
Also, $p \leq q \implies |u|^q \leq |u|^p \implies \int|u|^q \leq \int|u|^p \leq 1 \implies \int|u|^q \leq \left(\int|u|^q\right)^{\frac{q-2}{q}}$.

Now, we have,

$$\begin{aligned}\left(\int_{B(z,\theta R)}(|u|^q)^\kappa\right)^{\frac{1}{\kappa}} &\leq \left(\frac{13}{\tau R S_1}+\frac{4(q-1)^2\tau R}{S_1}\cdot K\right)\left(\int_{B(z,(\theta+\tau)R}|u|^q\right)^{\frac{q-2}{q}}\\ &\leq \frac{1}{\tau R S_1}\left(13+4(q-1)^2\tau^2R^2\cdot K\right)\left(\int_{B(z,\theta+\tau)R}|u|^q\right)^{\frac{q-2}{q}}\end{aligned}$$

For $\tau_o\in(0,1)$ fixed, take $\theta_0=1$, $\theta_{i+1}=\theta_i-\tau_{i+1}$, $q_i=p\kappa^i$, $\tau_i=\frac{\tau_0}{2^i}$. Then,

$$\begin{aligned}\left(\int_{B(z,\theta_{i+1}R)}|u|^{q_i\kappa}\right)^{\frac{1}{\kappa}} &\leq \frac{1}{\tau_{i+1}RS_1}(13+4(q_i-1)^2\tau_{i+1}^2R^2K)\left(\int_{B(z,\theta_iR)}|u|^{q_i}\right)^{\frac{q_i2}{q_i}}\\ &\leq \frac{C}{\tau_{i+1}R}\left(\int_{B(z,\theta_iR)}|u|^{q_i}\right)^{\frac{q_i-2}{q_i}}\\ \therefore\left(\int_{B(z,\theta_{i+1}R)}|u|^{p\kappa^{i+1}}\right)^{\frac{1}{\kappa}} &\leq 2^{i+1}\frac{C}{\tau_0R}\left(\int_{B(z,\theta_iR)}|u|^{p\kappa^i}\right)^{\frac{p\kappa^i-2}{p\kappa^i}}\\ &= 2^{i+1}\frac{C}{\tau_0R}\left[\left(\int_{B(z,\theta_iR)}|u|^{p\kappa^i}\right)^{\frac{1}{\kappa^i}}\right]^{\frac{p\kappa^i-2}{p}}\end{aligned}$$

If we denote $I_i=\left(\int_{B(z,\theta_iR)}|u|^{p\kappa^i}\right)^{\frac{1}{\kappa^i}}$,

$$\begin{aligned}I_{i+1} &\leq 2^{\frac{i+1}{\kappa^i}}\left(\frac{C}{\tau_0R}\right)^{\frac{1}{\kappa^i}}I_i^{\frac{p\kappa^i-2}{p\kappa^i}}\\ &\leq 2^{\sum_{k=0}^i\left(\frac{k+1}{\kappa^k}\cdot\frac{p\kappa^k-2}{p\kappa^k}\right)}\cdot\left(\frac{C}{\tau_0R}\right)^{\sum_{k=0}^i\left(\frac{1}{\kappa^k}\frac{p\kappa^k-2}{p\kappa^k}\right)}\cdot I_0^{\prod_{k=0}^i\left(\frac{p\kappa^k-2}{p\kappa^k}\right)}\end{aligned}$$

Take $i\to\infty$, $\sum_{k=0}^\infty\left(\frac{k+1}{\kappa^k}\cdot\frac{p\kappa^k-2}{p\kappa^k}\right)=n^2$, $\sum_{k=0}^\infty\left(\frac{1}{\kappa^k}\frac{p\kappa^k-2}{p\kappa^k}\right)=n$. Also, $\frac{p\kappa^k-2}{p\kappa^k}\in(0,1)$, $\forall k\in\mathbb{N}\implies\delta=\prod_{k=0}^\infty\left(\frac{p\kappa^k-2}{p\kappa^k}\right)\in(0,1)$.

That is,

$$\sup_{B(z,(1-\tau_0)R)}|u|^p\leq\frac{\tilde{C}}{\tau_0^nR^n}\left(\int_{B(z,R)}|u|^p\right)^\delta,\delta\in(0,1)$$

$\square$

We will now discuss the solvability of the Dirichlet problem at infinity under various assumption on $M$. The following is due to Cheng [Che].

THEOREM **1.3.** *Suppose $M^n$ complete, non-compact, Ricci curvatures bounded from below, $\lambda_1(M) > 0$ and $\inf_{x\in M}|B(x,r)| > 0$, $\forall r > 0$. Given a bounded smooth function $f$ on $M$ with $|\nabla f|$ bounded and $\int_M |\nabla f|^p < \infty$ for some $p \geq 2$. Then, $\exists u \in C^\infty(M)$ so that $\Delta(u+f) = 0$ on $M$, $\int_M |u|^p < \infty$ and $u(x) \to 0$ as $x \to \infty$.*

*Proof.* By adding a constant, we may assume that $f$ is positive. Theorem 1.1 implies that $\exists u \in C^\infty(M)$ such that $\Delta(u+f) = 0$ on $M$ and $\int_M |u|^p < \infty$. Since $u_R + f|_{\partial B(x_0,R)} = f \geq 0 \implies u_R + f \geq 0$ on $B(x_0, R)$. Also, $u+f$ is the limit of a subsequence of $\{u_R + f\}$ defined on $M$. Therefore, we have $u + f \geq$ and is bounded on $M$. This implies $\exists C_1$ constant such that $\dfrac{|\nabla(u+f)|}{u+f} \leq C_1$. Then, $|\nabla u|^2 \leq |\nabla(u+f)|^2 \leq C_1^2(u+f)^2 \leq C_2$, i.e., $|\nabla u| \leq C_3$. Now, we want to prove that $u(x) \to 0$ as $x \to \infty$.

Suppose $\exists x_n \in M$ such that $u(x_n) \not\to 0$ as $x_n \to \infty$, i.e., $\exists \varepsilon > 0$ such that $u(x_n) \geq \varepsilon > 0$ $\forall n \in \mathbb{N}$, then, for $x \in B(x_n, \frac{\varepsilon}{2C})$,

$$|u(x) - u(x_n)| \leq \int_0^l \left|\frac{d}{dt}u(\gamma(t))\right| \leq \int_0^l |\nabla u| \leq \int_0^l C = Cl \leq C \cdot \frac{\varepsilon}{2C} = \frac{\varepsilon}{2},$$

where $\gamma(t)$ is the minimal geodesic joining $x = \gamma(0)$ and $x_n = \gamma(l)$.

We then have,

$$\int_{B(x_n,\frac{\varepsilon}{2C})} |u(x)|^p \geq \left(\frac{\varepsilon}{2}\right)^p \left|B(x_n, \frac{\varepsilon}{2C})\right| \geq K\left(\frac{\varepsilon}{2}\right)^p > 0, K = \inf_{x\in M}\left|B(x, \frac{\varepsilon}{2C})\right| > 0.$$

WLOG, we can choose $\{x_n\}$ such that $B(x_n, \frac{\varepsilon}{2C}) \cap B(x_m, \frac{\varepsilon}{2C}) = \phi$. Thus,

$$\int_M |u|^p \geq \sum_{n=1}^{\infty} \int_{B(x_n,\frac{\varepsilon}{2C})} |u|^p \geq \sum_{n=1}^{\infty} K\left(\frac{\varepsilon}{2}\right)^p = \infty.$$

This contradiction shows that $u(x) \to 0$ as $x \to \infty$. □

Next, we have the following result by Anderson, Sullivan and Ancona.

THEOREM **1.4.** *Suppose $(M^n, d\tilde{s}^2)$ is a complete, simply connected Riemannian manifold such that the sectional curvatures $\widetilde{K_M}$ satisfies $-b^2 \leq \widetilde{K_M} \leq -a^2 < 0$.*

*Let $ds^2$ be another metric on $M$ uniformly equivalent to $d\tilde{s}^2$, i.e., $\exists A > 0$ such that $\frac{1}{A}ds^2 \leq d\tilde{s}^2 \leq Ads^2$. Then, for $f$ continuous on $M(\infty)$, $\exists u \in C^\infty(M) \cap C^0(M \cup M(\infty))$ such that*

$$\begin{cases} \Delta u = 0 \quad , \quad M \\ u = f \quad , \quad M(\infty) \end{cases} \tag{1.2}$$

*where $\Delta$ is the Laplacian operator of $(M, ds^2)$.*

*Proof.* WLOG, suppose $f$ is positive. Extend $f$ to $(M, d\tilde{s}^2)$ w.r.t. $x_0 \in M$ such that outside a compact set in $M$, it is radially constant. Then, with the curvature assumptions of $(M, d\tilde{s}^2)$, we can prove that $|\nabla_{d\tilde{s}^2} f|$ is of exponential decay and in $L^p$ for $p$ large enough (see Theorem 1.5). Now, $ds^2$ uniformly equivalent to $d\tilde{s}^2$ implies that $|\nabla_{ds^2} f|$ is also of exponential decay and is in $L^p$ for $p$ large enough. Also, the curvature assumptions on $(M, d\tilde{s}^2)$ implies $\lambda_1(M, d\tilde{s}^2) > 0$, which also implies that $\lambda_1(M, ds^2) > 0$. Theorem 1.1 now implies $\exists v \in C^\infty(M)$ such that $v$ is positive, $\Delta_{ds^2}(v + f) = 0$ on $(M, ds^2)$ and $v \in L^p(M, ds^2)$. Now, the curvature assumptions on $(M, d\tilde{s}^2)$ implies the existence of the Sobolev constant, i.e. $\exists S_1$ such that $S_1 \left(\int_M \phi^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_M |\nabla \phi|$, $\phi$ smooth on $(M, d\tilde{s}^2)$. $ds^2$ uniformly equivalent to $d\tilde{s}^2$ implies the existence of $S_1$ on $(M, ds^2)$. Hence, applying Theorem 1.2 on $(M, ds^2)$ shows that $v(x) \to 0$ as $x \to \infty$. Write $u = v + f$. Then, $\Delta u = 0$, $u(x) \to f$ as $x \to \infty$ and $u \in L^p(M, ds^2)$. □

We will now generalize the problem by assuming a more general curvature assumption.

LEMMA **1.3.** *Let $M$, $M_0$ be Riemannian manifolds, $\gamma$, $\gamma_0 : [0, l] \to M$, $M_0$ be normalized geodesics. For $t \in [0, l]$, $\mathbb{X} \in M_{\gamma(t)}$, $\mathbb{X}_0 \in (M_0)_{\gamma_0(t)}$, let $K(\sigma)$, $K_0(\sigma_0)$ denote the sectional curvatures of the plane sections spanned by $\mathbb{X}$, $\gamma'(t)$ and $\mathbb{X}_0$, $\gamma_0'(t)$ respectively. Assume that $K(\sigma)$, $K_0(\sigma_0) \leq 0$, $|K_0(\sigma_0) \leq C|K(\sigma)|$, $C \geq 1$ a constant independent of $t$, $\sigma$, $\sigma_0$. Let $V$, $V_0$ be Jacobi fields along $\gamma$, $\gamma_0$ such that*

$V(0) = V_0(0) = 0$, $V'(0) \perp \gamma'(0)$, $V_0(0) \perp \gamma_0'(0)$ *and* $\|V'(0)\| = \|V_0'(0)\|$. *Then,* $\dfrac{\|V(t)\|^C}{\|V_0(t)\|}$ *is an increasing function of* $t$.

*Proof.* Fix $t_1 \in [0, l)$, define $W_{t_1}(t) = \frac{V(t)}{\|V(t_1)\|}$, $(W_0)_{t_1} = \frac{V_0(t)}{\|V_0(t_1)\|}$. Consider

$$\begin{aligned}
\frac{d}{dt}\left(\log \|W\|^2\right) &= \frac{d}{dt}\left[\log \frac{< V(t), V(t) >}{\|V(t_1\|^2}\right] = \frac{2 < V(t), V'(t) >}{< V(t), V(t) >} \\
\frac{d}{dt}\left(\log \|W_0\|^2\right) &= \frac{2 < V_0(t), V_0'(t) >}{< V_0(t), V_0(t) >}
\end{aligned}$$

Now, $\|W(t_1)\| = 1$, consider at $t_1$,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t_1}\left(\log \|W\|^2\right) &= \left.\frac{2 < V, V' >}{< V, V >}\right|_{t_1} \\
&= 2 < W, W' >|_{t_1} = 2\int_o^{t_1} < W, W' >' \\
&= 2\int_o^{t_1} < W', W' > + < W, W'' > \\
&= 2\int_0^{t_1} < W', W' > - K(\sigma) < W, W > \\
&= 2\int_0^{t_1} < W', W' > - K(\sigma)\|W\|^2 \\
\left.\frac{d}{dt}\right|_{t_1}\left(\log \|W_0\|^2\right) &= 2\int_0^{t_1} < W_0', W_0' > - K_0(\sigma_0)\|W_0\|^2
\end{aligned}$$

Let $\{e_1, ..., e_n\}$, $\{\hat{e_1}, ..., \hat{e_n}\}$ be orthonormal frames along $\gamma$, $\gamma_0$ respectively. Suppose $W(t) = \sum W^i(t)e_i$. Let $\hat{W_0}(t) = \sum W^i(t)\hat{e_i}$ along $\gamma_0$. Then, $< \hat{W_0}, \hat{W_0} >=< W, W >$, $< \hat{W_0}', \hat{W_0}' >=< W', W' >$.

Let $\hat{\sigma}$ be the plane section spanned by $\hat{W}_0$ and $\gamma_o'$. Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\log\frac{\|W\|^{2C}}{\|W_0\|^2}\right)\\
= \;& C\int_0^{t_1}(<W',W'> - K(\sigma)\|W\|^2) - \int_0^{t_1}(<W_0',W_0'> - K_0(\sigma_0)\|W_0\|^2)\\
= \;& C\int_0^{t_1}\left(<\hat{W_0}',\hat{W_0}'> - K_0(\sigma_0)\|\hat{W}_0\|^2\right) + C\int_0^{t_1}\left(K_0(\hat{\sigma})\|\hat{W}_0\|^2 - K(\sigma)\|W\|^2\right)\\
&-\int_0^{t_1}(<W_0',W_0'> - K_0(\sigma_0)\|W_0\|^2)\\
\geq \;& (C-1)\int_0^{t_1}\left(<\hat{W_0}',\hat{W_0}'> - K_0(\sigma_0)\|\hat{W}_0\|^2\right)\\
&+C\int_0^{t_1}\left(K_0(\hat{\sigma})\|\hat{W}_0\|^2 - K(\sigma)\|W\|^2\right)\\
\geq \;& (C-1)\int_0^{t_1}\left(-K_0(\sigma_0)\|\hat{W}_0\|^2\right) + C\int_0^{t_1}\left(K_0(\hat{\sigma})\|\hat{W}_0\|^2 - K(\sigma)\|W\|^2\right)\\
= \;& \int_0^{t_1}(K_0(\sigma_0) - CK(\sigma))\,\|W\|^2\\
\geq \;& 0
\end{aligned}$$

since $\|\hat{W}_0\| = \|W\|^2$. $t_1$ arbitrary $\Longrightarrow$ done.

□

We have the following generalization of the result of Anderson, Sullivan and Ancona by Cheng [Che].

THEOREM **1.5.** *Suppose $M$ complete, simply connected with non-positive curvature and $\lambda_1(M) > 0$. Assume that $\exists x_0 \in M$, constant $C \geq 1$ such that $\forall x \in M$, we have $|K(\sigma)| \leq C|K(\sigma')|$, where $\sigma$, $\sigma'$ are plane sections at $x$ containing the tangent vector of the geodesic joining $x_0$ to $x$. Then, for $f$ continuous on $M(\infty)$, $\exists u \in C^\infty(M) \cap C^0(M \cup M(\infty))$ such that*

$$\begin{cases} \Delta u = 0 \quad , & M \\ u = f \quad , & M(\infty) \end{cases} \tag{1.3}$$

*Proof.* First assume $f$ is smooth. Extend $f$ to a smooth function on $M$ such that for some $x_0 \in M$, $f$ is radially constant w.r.t. $x_0$ outside $B(x_0, \frac{1}{2})$. We want

to prove that $|\nabla f|$ bounded on $M$ and $|\nabla f| \in L^p(M)$. Let $J_M$, $J_{\mathbb{R}^n}$ be Jacobi fields on $M$ and $\mathbb{R}^n$ respectively such that $J_M(0) = J_{\mathbb{R}^n}(0) = 0$, $J'_M(0) \perp J'_{\mathbb{R}^n}(0)$, $|J'_M(0)| = |J'_{\mathbb{R}^n}(0)|$. Rauch comparison Theorem then implies $|J_M| \geq |J_{\mathbb{R}^n}|$.
Define $j(x) = inf\{\|V(r)\| : V$ is a Jacobi field along $exp_p rv = \gamma(r)$, $V(0) = 0$, $|V'(0)| = 1$, $V'(0) \perp \gamma'(0)\}$. First, we will show that $|\nabla f| \leq \dfrac{A}{j(x)}$: $f$ radially constant, i.e., $\forall i$, $\dfrac{\partial f}{\partial \theta^i} \leq \tilde{C}$. Now, $|\nabla f|^2 = g^{ij}\left(\dfrac{\partial f}{\partial \theta^i}\dfrac{\partial f}{\partial \theta^j}\right)$, $i, j = 1, ..., n-1$.
Choose $\dfrac{\partial}{\partial \theta^1} = \nabla f$, $|\nabla f|^2(x) = g^{11}\left(\dfrac{\partial f}{\partial \theta^1}\right)^2 \leq \dfrac{A}{g_{11}(x)}$, $x = (r, \theta_1, ..., \theta_{n-1})$ in geodesic polar coordinates. Now,

$$\begin{aligned} g_{11} &= <(dexp_p)_{rv}\frac{\partial}{\partial\theta^1}, (dexp_p)_{rv}\frac{\partial}{\partial\theta^1}> \\ &= \frac{1}{r^2}\left|\frac{\partial}{\partial\theta^1}\right|^2\left|(dexp_p)_{rv}\, r\frac{\frac{\partial}{\partial\theta^1}}{\left|\frac{\partial}{\partial\theta^1}\right|}\right|^2 \\ &= \left|(dexp_p)_{rv}\, r\frac{\frac{\partial}{\partial\theta^1}}{\left|\frac{\partial}{\partial\theta^1}\right|}\right|^2 \geq j^2(x) \end{aligned}$$

Hence, $|\nabla f| \leq \dfrac{A}{j(x)}$. Therefore, comparing with Jacobi fields on $\mathbb{R}^n$, we have $|\nabla f| \leq \dfrac{A}{j(x)} \leq \dfrac{A}{C_2 t}$. Hence, $|\nabla f|$ decays uniformly to zero.
Next, we want to prove that $|\nabla f| \in L^p$. Consider for $x \in M$, let $\gamma(t)$ be the unit speed geodesic joining $\gamma(0) = x_0$, $\gamma(r) = x$. Define $F = \{V(t) : V(t)$ is a Jacobi field on $\gamma$, $V(0) = 0$, $|V'(0)| = 1$, $V'(0) \perp \gamma'(0)\}$, $J(x) = \sup_{V\in F}|V(r)|$, $j(x) = \inf_{V\in F}|V(r)|$.

Consider the Jacobian of the exponential map:

$$\begin{aligned} \psi(t) &= \frac{1}{t^{n-1}}|J_1(t)\wedge ... \wedge J_{n-1}(t)| \\ &= \frac{1}{t^{n-1}}det\left(< J_i, J_j > (t)\right)_{ij} \\ &= \frac{1}{t^{n-1}}det(J_1, ..., J_{n-1}) \\ &= \frac{1}{t^{n-1}}|J_1|...|J_{n-1}| \leq \frac{1}{t^{n-1}}J_{n-1}(x) \end{aligned}$$

$$\text{Hence, } \int_M g\,dV \leq \int_M gJ^{n-1}(x)\,dr\,d\theta \quad \text{for } g \text{ integrable.}$$

Using Lemma 1.3, we have $\dfrac{j^C(x)}{J(x)}$ is an increasing function of the distance function $\rho(x, x_0)$. Then, for $x$ outside $B(x_0, \frac{1}{2})$, we have,

$$\frac{j^C(x)}{J(x)} \geq C_1 = \sup_{\tilde{x} \in \partial B(x_0, \frac{1}{2})} \frac{j^C(\tilde{x})}{J(\tilde{x})} \implies J(x) \leq \frac{j^C(x)}{C_1}$$

Then,

$$\frac{J^{n-1}(x)}{j^p(x)} \leq C_1^{1-n} j^{C(n-1)-p}(x)$$

$$\begin{aligned}\int |\nabla f|^p dV &\leq \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{A^p}{j^p(r,\theta)} \cdot J^{n-1}(r,\theta) d\theta dr \\ &\leq \tilde{C} \int_0^\infty \int_{\mathbb{S}^{n-1}} j^{C(n-1)-p}(r,\theta) d\theta dr\end{aligned}$$

Since we have $j(r, \theta) \geq C_2 r$, then, choose $p$ large enough such that $C(n-1)-p \leq -2 < 0$. Then,

$$\int |\nabla f|^p dV \leq \tilde{C} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1}{r^2} d\theta dr < \infty$$

Hence, $|\nabla f| \in L^p$.

Then, Theorem 1.1 implies $\exists u \in C^\infty(M)$, $\int_M |u|^p < \infty$ such that $\Delta(u + f) = 0$, $|\nabla f|$ decays uniformly to zero $\implies$ $|\nabla f|$ bounded. $K_M \leq 0 \implies \exists S_1$ Sobolev constant. By Theorem 1.2, we have $u(x) \to 0$ as $x \to \infty$.

Now, suppose $f_n$ smooth on $M$ such that $f_n \to f$. Then, $\exists u_n \in C^\infty(M)$ such that

$$\begin{cases} \Delta u_n = 0 & , \quad M \\ u_n = f_n & , \quad M(\infty) \end{cases} \tag{1.4}$$

By Arzela Ascoli's Theorem, $\exists u$ such that $u_n \to u$, passing to subsequence if necessary. Using maximum principle on harmonic functions,

$$\sup_{x \in M} |u_n - u_m|(x) \leq \sup_{x \in M(\infty)} |f_n - f_m|(x)$$

Taking $n \to \infty$,

$$\sup_{x \in M} |u - u_m|(x) \leq \sup_{x \in M(\infty)} |f - f_m|(x)$$

Take $x \to M(\infty)$,

$$|u - u_m(x)| \to |u - f_m|(x) \leq \sup_{x \in M(\infty)} |f - f_m|(x)$$

Now take $m \to \infty$, *l.h.s.* $\to |u - f|(x)$, $x \in M(\infty)$, *r.h.s.* $\to 0$. Hence $u = f$ on $M(\infty)$. Using Schauder Estimates,

$$\begin{cases} \Delta u = 0 \quad , \quad M \\ u = f \quad , \quad M(\infty) \end{cases} \tag{1.5}$$

$\square$

Corollary **1.1.** *Suppose that $(M, ds^2)$ is the same as in Theorem 1.5 and that $(M, ds_1^2)$ is uniformly equivalent to $(M, ds^2)$. Then, on $(M, ds_1^2)$, the Dirichlet problem at infinity can be solved.*

# Chapter 2

# The Martin Boundary

The Martin boundary was first introduced by Robert S. Martin during the early 40's. In [Mar], Martin developed an abstract boundary for $M = \mathbb{R}^3$, the Martin boundary $\widetilde{M}$, and an integral representation formula for a non-negative harmonic function $u$. It states that there exists a measure $\mu$ such that

$$u(x) = \int_{\widetilde{M}} K(x,y)d\mu(y)$$

This $\mu$, determined by $u$ is unique iff $u$ is minimal, i.e., if $v$ is also a non-negative harmonic function such that $v < u$, then $v = ku$, where $k$ is a constant. In the following sections, we will follow the original idea of Martin to develop this representation formula.

Let $M$ be a complete non-compact manifold. In this chapter, we always assume $M$ has a minimal positive Green's function $G(x,y)$, for $x$, $y \in M$:

(i) $G(x,y) = G_y(x) = G(y,x)$ is harmonic in $x$ on $M \backslash \{y\}$,

(ii) $G(x,y) \geq 0$,

(iii) as $y \to x$, $\rho = \rho(x,y)$ the distance between $x$, $y$, $G_x(y) \sim \rho^{2-n}$ for $n > 2$, $\log \frac{1}{\rho}$ for $n = 2$

(iv) if $G_R(x,y)$ is the positive Green's function defined on $\overline{B(R)}$ with $G_R(x,y) = 0$ on $\partial B(R)$, $G(x,y) = \lim_{R\to\infty} G_R(x,y)$.

Now, for a fixed $y_0 \in M$, we define a function $K(x,y)$, $x$, $y$ in $M$ to be

$$K(x,y) = \begin{cases} \frac{G(x,y)}{G(x,y_0)} & x \neq y_0 \\ 0 & x = y_0, y \neq y_0 \\ 1 & x = y = y_0 \end{cases}$$

Consider a sequence of points $\{y_n\}$ in $M$, we said that it is fundamental iff the corresponding sequence of functions $K(x, y_n)$ converges to a harmonic function in $M$. Thus, if $\{y_n\}$ has no accumulation point in $M$, we can always find a subsequence of it such that the subsequence found is fundamental.

DEFINITION **2.1.** *The Martin Boundary $\widetilde{M}$ of $M$ is defined to be all equivalent classes of fundamental sequences which have no limit points in $M$.*

## 2.1 THE MARTIN METRIC

In this section, we will define a metric $\rho$ on $M \cup \widetilde{M}$ such that $(M \cup \widetilde{M}, \rho)$ is compact.
For $y_1$, $y_2 \in \overline{M} = M \cup \widetilde{M}$, define

$$\rho(y_1, y_2) = \int_\Sigma \frac{|K(x,y_1) - K(x,y_2)|}{1 + |K(x,y_1) - K(x,y_2)|} dx,$$

where $\Sigma$ is some compact set in $M$. For $y \in \widetilde{M}$, $K(x,y)$ is taken to be the limit of the corresponding sequence of harmonic functions of any fundamental sequence converging to $y$.

PROPOSITION **2.1.** *$(\overline{M}, \rho)$ is a compact metric space.*

*Proof.* We want to prove that for a sequence $\{y_n\} \in \overline{M}$, we can always find a subsequence which converges in $\overline{M}$. First, suppose that all $y_n$ are in $M$.
Case 1: There exists an accumulation point $y$ of $\{y_n\}$ in $M$, then there exists a subsequence of $\{y_n\}$, say, $\{y_{n_k}\}$ converges to $y$ and hence $|K(x, y_{n_k}) - K(x,y)| \to 0$ as $k \to \infty$. Now, since $\dfrac{|K(x,y_{n_k}) - K(x,y)|}{1 + |K(x,y_{n_k}) - K(x,y)|} \leq 1$ for all $k$, we

have $\rho(y_{n_k}, y) \to 0$ as $k \to \infty$.

Case 2: $\{y_n\}$ has no accumulation point in $M$. Then, we can always find a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $K(x, y_{n_k})$ converges to a harmonic function, say, $K(x, \xi)$ in $M$. Hence,

$$|K(x, y_{n_k}) - K(x, \xi)| = \left| \frac{G(x, y_{n_k})}{G(x_0, y_{n_k})} - \lim_{i \to \infty} \frac{G(x, y'_{n_k})}{G(x_0, y'_{n_k})} \right| \to 0$$

as $k \to \infty$. That is, $\rho(y_{n_k}, \xi) \to 0$.

Now, suppose all $y_n \in \widetilde{M}$. For each $y_n$, there exists a fundamental sequence converging to $y_n$, take $y'_n$ from it such that $\rho(y_n, y'_n) \leq \frac{1}{n}$. Then, there exists a subsequence of $\{y'_n\}$ such that it is fundamental, therefore, $y'_n \to y \in \widetilde{M}$,

$$\rho(y_n, y) \leq \rho(y_n, y'_n) + \rho(y'_n, y) \leq \frac{1}{n} + \rho(y'_n, y) \to 0$$

as $n \to \infty$. □

PROPOSITION **2.2.** *$(M, \rho)$ coincide with the original topology of $M$.*

*Proof.* We want to prove that a sequence converging in $(M, \rho)$ also converges in $(\overline{M}, \rho)$ and vice versa. First of all, if $y_n \to y$, $y_n$, $y \in M$, clearly we have $\rho(y_n, y) \to 0$.

Now suppose $y_n \in M$, $\rho(y_n, y) \to 0$, $y \in M$ but $y_n \not\to y$. Then, either

(i) there exists $y' \in M$ such that $y_{n_k} \to y' \neq y$ for some subsequence of $\{y_{n_k}\}$. Therefore, $\rho(y_{n_k}, y') \to 0$.

(ii) there exists a fundamental subsequence of $\{y_n\}$, say, $\{y_{n_k}\}$. Therefore, $\rho(y_{n_k}, y') \to 0$ for some $y' \in \widetilde{M}$. □

## 2.2 The Representation Formula

We will now prove the representation formula for positive harmonic functions, we need the followings first:

THEOREM **2.1.** *Let $u(x)$ be positive and superharmonic in $M$. Let $\sigma$ be a closed subset of $M$. There exists a function $u^*_\sigma(x)$ uniquely defined in $M$ such that*

*(a) $u^*_\sigma(x)$ is superharmonic in $M$,*

*(b) $u^*_\sigma(x) = u(x)$ on $\sigma$,*

*(c) In $M\backslash\sigma$, $u^*_\sigma(x)$ is equal to the harmonic function $\phi$ on $M\backslash\sigma$ such that $\phi = u$ at the boundary point of $M\backslash\sigma$ lying in $M$ and 0 at other boundary points(infinity).*

For this $u^*_\sigma$, we have the following properties:

THEOREM **2.2.** *(a) $u(x) \geq u^*_\sigma(x) \geq 0$ for all $x \in M$,*

*(b) If $u(x) \geq v(x)$, then $u^*_\sigma(x) \geq v^*_\sigma(x)$ for all $x \in M$,*

*(c) $(u+v)^*_\sigma(x) = u^*_\sigma(x) + v^*_\sigma(x)$,*

*(d) $(c \cdot u)^*_\sigma(x) = c \cdot u^*_\sigma(x)$, $c > 0$ constant,*

*(e) If $u_n(x) \to u(x)$ and there exists a superharmonic majorant $U(x)$ to $u_n(x)$, then $(u_n)^*_\sigma(x) \to u^*_\sigma(x)$. Lacking the $U(x)$, we still have $\underline{\lim}(u_n)^*_\sigma(x) \geq u^*_\sigma(x)$,*

*(f) If $\sigma \subseteq \tau$, then $(u^*_\tau)^*_\sigma(x) = (u^*_\sigma)^*_\tau(x) = u^*_\sigma(x)$,*

*(g) If $\sigma \subseteq \tau$, then $u^*_\sigma(x) \leq u^*_\tau(x)$. If $\sigma_n \uparrow \sigma$, then $u^*_{\sigma_n}(x) \uparrow u^*_\sigma(x)$,*

*(h) $u^*_{\sigma+\tau}(x) \leq u^*_\sigma(x) + u^*_\tau(x)$.*

Detailed proofs of theorem 2.1 and 2.2 can be found in [Mar].

THEOREM **2.3.** *If $u(x)$ non-negative, superharmonic and continuous in $M$, $\sigma$ a bounded closed subset of $M$, then,*

$$u^*_\sigma(x) = \int_\sigma K(x,y)d\nu_\sigma(y)$$

*where $\nu_\sigma$ is a positive finite Borel measure over $\sigma$.*

The proof of theorem 2.3 mainly used Riesz's result in a form due to Frostman [Fro], i.e. the functional

$$J(\nu) = \frac{1}{2}\int_\sigma\int_\sigma K(x,y)d\nu_\sigma(y)d\nu_\sigma(x) - \int_\sigma u(x)d\nu(x)$$

is minimized by a unique $\sigma_n u$ among all non-negative measures $\nu$ whose total mass in $\sigma$.

Now, let $G \subseteq \overline{M}$, denote by $[G]$ the intersection of $M$ with the $\rho$ closure of $G$.

DEFINITION **2.2.** *Let $u(x)$ be a non-negative harmonic function in $M$, $A$ be a closed subset of $\widetilde{M}$. The function $u_A(x)$ is defined to be the greatest lower bound of $u^*_{[G]}(x)$ as $G$ ranges over all open sets containing $A$.*

In fact we can show that $u_A(x)$ is the limit of $u^*_{[G_n]}(x)$ for a descending sequence of $G_n$:

LEMMA **2.1.** *Let $G_1$, $G_2$,... a descending sequence of open sets which contain $A$ and $\cap G_n = A$. Then, $u^*_{[G_n]}(x) \downarrow u_A(x)$.*

*Proof.* By definition, $u^*_{[G_n]}(x)$ is a decreasing function for $n$. Since $u^*_{[G_n]}(x)$ is non-negative harmonic on $M \backslash [G_n]$, $\cap_{n \in \mathbb{N}}[G_n] = \phi$, $v(x) = \lim u^*_{[G_n]}(x)$ is a non-negative harmonic function on $M$. Also, $v(x) \geq u_A(x)$. Consider for $\varepsilon > 0$, $\exists$ open set $G \supseteq A$ such that $u^*_{[G]}(x) \leq u_A(x) + \varepsilon$.

We claim that $G_n \subset G$ for $n$ large. Now, $\phi = A \backslash G = \left(\cap \overline{G_n}\right) \backslash G = \cap \left(\overline{G_n} \backslash G\right)$. Therefore, $\overline{M} = \cup \left(\overline{G_n} \backslash G\right)^c$, $\overline{M}$ compact, $\left(\overline{G_n} \backslash G\right)^c$ open. Hence, there exists finite subcover of $\overline{M}$ such that $\overline{M} = \cup_{n=1}^{N} \left(\overline{G_n} \backslash G\right)^c$. That is, $\phi = \cap_{n=1}^{N} \left(\overline{G_n} \backslash G\right) = \left(\cap_{n=1}^{N} \overline{G_n}\right) \backslash G = \overline{G_n} \backslash G$. Therefore, for $n \geq N$, $G_n \subset G$.

Then, $u^*_{[G]}(x) \geq v(x)$, i.e., $v(x) \leq u^*_{[G]}(x) \leq u_A(x) + \varepsilon$. Take $\varepsilon \to 0$, we have $v(x) = u_A(x)$. □

To prove the representation theorem, we need the following properties of $u_A(x)$:

THEOREM **2.4.** *The function $u_A(x)$ is non-negative harmonic in $M$ and satisfies*

*(a) $u(x) \geq u_A(x)$ for all $x \in M$,*

*(b) If $u(x) \geq v(x)$ for all $x \in M$, then, $u_A(x) \geq v_A(x)$,*

*(c) $(u + v)_A(x) = u_A(x) + v_A(x)$,*

*(d) $(c \cdot u)_A(x) = c \cdot u_A(x)$, $c > 0$ constant,*

*(e)* $u_{\widetilde{M}}(x) = u(x)$,

*(f) If* $A \supseteq B$, *then* $(u_B)_A(x) = u_B(x)$,

*(g) If* $A \supseteq B$, *then* $u_A(x) \geq u_B(x)$. *If* $A_n \downarrow A$, *then* $u_{A_n}(x) \downarrow u_A(x)$,

*(h)* $u_{A+B}(x) \leq u_A(x) + u_B(x)$.

*Proof.* (a) by definition, $u_A(x) = \inf u^*_{[G]}(x) \leq u(x)$,

(b) for $G$ open in $\overline{M}$ containing $A$, we have $u^*_{[G]}(x) \geq v^*_{[G]}(x) \geq v_A(x)$. Taking the infimum of $G$, we have $u_A(x) \geq v_A(x)$,

(c) take $G_n \downarrow A$ open such that $\cap \overline{G_n} = A$,

$$(u+v)_A(x) = \lim (u+v)^*_{[G_n]}(x) = \lim \left(u^*_{[G_n]}(x) + v^*_{[G_n]}(x)\right) = u_A(x) + v_A(x),$$

(d) $(c \cdot u)_A(x) = \inf (c \cdot u)^*_{[G]}(x) = \inf c \cdot u^*_{[G]}(x) = c \inf u^*_{[G]} = c \cdot u_A(x)$, $c > 0$,

(e) suppose $G \supseteq \widetilde{M}$ open. Then, $\overline{M} \backslash G \subseteq M$ is $\rho$ closed and bounded. Therefore, $M \backslash [G]$ is bounded, open and interior to $M$. Since $u(x)$ is harmonic, $u^*_{[G]}(x) = u(x)$ in $M \backslash [G]$. Take $G \downarrow \widetilde{M}$, left hand side tends to $u_{\widetilde{M}}(x)$,

(f) first of all, we have $u_B \geq (u_B)_B$. Suppose $H_n \supset B$, $\downarrow$, $\cap \overline{H_n} = B$. Then, $(u_B)_B(x) = (u_B)^*_{[H_n]}(x) - \varepsilon_n(x) = (u^*_{[H_n]})^*_{[H_n]}(x) - (\eta_n)^*_{[H_n]}(x) - \varepsilon_n(x) \geq u_B(x) - \eta_n(x) - \varepsilon_n(x)$. Taking $n \to \infty$, we get $(u_B)_B(x) \geq u_B(x)$ hence $(u_B)_B(x) = u_B(x)$. Finally, $u_B(x) = (u_B)_B(x) \leq (u_B)_A(x)$,

(g) let $A \supseteq B$ be closed subsets of $\widetilde{M}$, $G \supseteq A$, $H \supseteq B$ open in $\overline{M}$, then, $u_A(x) = \inf u^*_{[G]}(x) = \inf u^*_{[H]}(x) = u_B(x)$. Now, let $A_n \downarrow A$, $\cap A_n = A$, $G \supset A$ open, then, $\phi = A \backslash G = (\cap A_n) \backslash G = \cap (A_n \backslash G)$. Hence, $\overline{M} = \cup (A_n \backslash G)^c = \cup_{n=1}^N (A_n \backslash G)^c$. Therefore, $\phi = \cap_{n=1}^N (A_n \backslash G) = A_N \backslash G$, i.e., $A_n \subset G$ for $n \geq N$. Therefore $v(x) = \lim u_{A_n}(x) \leq u^*_{[G]}(x) \leq u_A(x) + \varepsilon$, i.e., $v(x) \leq u_A(x)$. Since $u_A(x) \leq u_{A_n}(x)$ for all $n$, we have $v(x) = u_A(x)$,

(h) let $G \supset A$, $H \supset B$ open, $K = G \cup H$, $[K] = [G] \cup [H]$. Since $u^*_{[G]}(x) + u^*_{[H]}(x) \geq u^*_{[K]}(x)$. by taking $G \downarrow A$, $H \downarrow B$, $K \downarrow A \cup B$, we have $u_A(x) + u_B(x) \geq u_{A \cup B}(x)$,

□

THEOREM **2.5.** *If* $u(x)$ *is a non-negative harmonic function in* $M$, $A \subseteq \widetilde{M}$

*closed, then, there exists measure* $\mu_A$ *on* $A$ *such that*

$$u_A(x) = \int_A K(x,y)d\mu_A(y)$$

*for* $x \in M$.

*Proof.* Let $G \supseteq A$ open, $x_0 \notin \overline{G}$, $\overline{G} = \rho$ closure of $G$, $\sigma$ closed in $[G]$. Then, $G(y, x_0)$ positive continuous in $\sigma$.

$$\begin{aligned} u_\sigma^*(x) &= \int_\sigma G(y,x)d\nu_\sigma(y) = \int_\sigma K(x,y)G(y,x_0)d\nu_\sigma(y) = \int_\sigma K(x,y)d\mu_\sigma(y) \\ &= \int_{\overline{G}} K(x,y)d\mu_\sigma(y), \end{aligned}$$

considering $\mu_\sigma$ to be a measure over $\overline{G}$ with total mass in $\sigma$ and equals $u_\sigma^*(x_0)$. Now, let $\sigma_n \uparrow [G]$, then, $u_{\sigma_n}^*(x) \uparrow u_{[G]}^*(x)$. Since the total mass of $\mu_{\sigma_n}$ is bounded by $u(x_0)$, there exists a weak limit, say, $\mu_G$ of $\{u_{\sigma_n}\}$ with total mass in $\overline{G}$. Therefore,

$$u_{[G]}^*(x) = \int_{\overline{G}} K(x,y)d\mu_G(y)$$

Then, let $G_n \downarrow A$, $\mu_{G_n}$ converges weakly to a measure $\mu_A$. Hence, $u_A(x) = \int_A K(x,y)d\mu_A(y)$ with total mass of $\mu_A$ is in $A$, equals $u_A(x_0)$. □

THEOREM **2.6.** *If* $u(x)$ *non-negative harmonic in* $M$, *then* $\exists$ *measure* $\mu$ *over* $\widetilde{M}$ *such that*

$$u(x) = \int_{\widetilde{M}} K(x,y)d\mu(y)$$

*for* $x \in M$.

*Proof.* Let $G \supseteq \widetilde{M}$, $x_0 \notin G$.

$$u_{[G]}^*(x) = \int_G K(x,y)d\mu_G(y)$$

with total mass of $\mu_G$ equals $u_G(x_0) \leq u(x_0)$. Then, let $G_n \downarrow \widetilde{M}$, $\mu_{G_n} \rightharpoonup \mu$ a measure over $\widetilde{M}$. Since $u_{G_n}(x) \downarrow u_{\widetilde{M}}(x) = u(x)$, therefore,

$$u(x) = \int_{\widetilde{M}} K(x,y)d\mu(y)$$

with total mass of $\mu$ equals $u(x_0)$. □

## 2.3 Uniqueness of Representation

In this section, we will introduce the canonical representation. We will also show that when a representation is canonical, it is in fact unique. By a minimal harmonic function, say, $f$ in $M$ we mean that if $g$ is also a harmonic function in $M$ satisfying $g \leq f$, then $g$ is a constant multiple of $f$.

Lemma **2.2.** *Suppose $u(x)$ is positive harmonic and minimal in $M$. Let $A$ be any $\rho$ Borel subset of $\widetilde{M}$. If for $x \in M$, $u(x) \geq \int_A K(x,y)d\mu(y) > 0$, then $u(x) = u(x_0)K(s,x)$, $s \in A$.*

*Proof.* Since $u(x_0) \geq \int_A K(x_0,y)d\mu(y) = \mu(A) > 0$, $\therefore \exists A_1 \subset\subset A$ such that $\mu(A_1) > 0$. Choose $A_2$ in the finite cover of $A_1$ such that $\mu(A_2) > 0$. Repeating this step inductively, we can find a nested sequence $A_1 \supset A_2 \supset \ldots$ such that the diameter goes down to zero. Take $s \in \cap_{n\in\mathbb{N}} A_n$, since $\mu(A_n) > 0$, $\int_{A_n} K(x,y)d\mu_{A_n}(y) = c_n u(x)$ is harmonic, $u$ minimal implies $\int_{A_n} K(x,y)d\mu_{A_n}(y) = c_n u(x)$, $c_n$ a constant. Let $\mu_n = \frac{1}{c_n}\mu_{A_n}$. Then, $u(x) = \int_{A_n} K(x,y)d\mu_n(y)$ with total mass of $\mu_n$ equals $u(x_0$. Then, there exists subsequence of $\{\mu_n\}$ converging weakly. Therefore, letting $n \to \infty$, $u(x) = u(x_0)K(s,x)$. □

Corollary **2.1.** *If $u$ is minimal positive harmonic in $M$, $u$ is a positive multiple of $K(s,x)$, $s \in \widetilde{M}$.*

Corollary **2.2.** *If $K(s,x)$ minimal, $A$ closed subset of $\widetilde{M}$ such that $K_A(s,x)$ positive, then $s \in A$.*

*Proof.* $K(s,x) \geq K_A(s,x) = \int_A K(x,y)d\mu_A(y) > 0$. By lemma, $K(s,x) = K(s_1,x_0)K(s_1,x)$ for some $s_1 \in A \implies K(s,x) = K(s_1,x) \implies s = s_1 \in A$. □

Definition **2.3.** *For $s \in \widetilde{M}$, denote $\psi(s) = K_{\{s\}}(s,x_0)$.*

Theorem **2.7.** $\psi(s) = 1$ *iff* $K(s,x)$ *minimal and* 0 *iff* $K(s,x)$ *not minimal.*

*Proof.* $K_{\{s\}}(s,x) = \int_{\{s\}} K(x,y)d\mu_s(y) = K_{\{s\}}(s,x_0)K(s,x) = \psi(s)K(s,x)$.
l.h.s.$=\left(K_{\{s\}}\right)_{\{s\}}(s,x) = (\psi(s)K(s,x))_{\{s\}}(s,x) = \psi(s)K_{\{s\}}(s,x)$. Put $x = x_0$, $\psi(s) = (\psi(s))^2$ hence $\psi(s) = 0$ or 1.

Now, suppose $\psi(s) = 1$, let $u(x) \le K(s,x)$, $u(x)$ positive harmonic function. Then $v(x) = K(s,x) - u(x)$ is also positive harmonic. We have, $u(x) \ge u_{\{s\}}(x)$, $v(x) \ge v_{\{s\}}(x)$. $K(s,x) = u(x) + v(x) \ge u_{\{s\}}(x) + v_{\{s\}}(x) = K_{\{s\}}(s,x) = \psi(x)K(s,x) = K(s,x)$. Therefore, $u(x) = u_{\{s\}}(x) = u_{\{s\}}(x_0)K(s,x)$ and hence $K(s,x)$ minimal.

Conversely, assume $K(s,x)$ is minimal. Let $A$ be a closed subset of $\widetilde{M}$, $s$ contained in the interior of $A$. Let $B = \overline{\left(\widetilde{M}\backslash A\right)}$, if $K_B(s,x) > 0$, previous corollary implies that $s \in B$. Therefore, $K_B(s,x) = 0$. Then, $K(s,x) = K_{\widetilde{M}}(s,x) = K_{A\cup B}(s,x) \le K_A(s,x) + K_B(s,x) = K_A(s,x) \le K(s,x)$. Hence, $K(s,x) = K_A(s,x)$. Let $A_n \in \widetilde{M}$ such that $\rho(A_n, s) < \frac{1}{n}$, $s \in A_n$ for all $n$. Then $A_n \downarrow \{s\}$, $K(s,x) = K_{\{s\}}(s,x)$. Put $x = x_0$, $\psi(s) = K(s,x_0) = 1$. □

DEFINITION **2.4.** $M_1 = \{s \in \widetilde{M} : \psi(s) = 1\}$, $M_0 = \{s \in \widetilde{M} : \psi(s) = 0\}$.

DEFINITION **2.5.** *A measure $\mu$ over $\widetilde{M}$ is called canonical iff $\mu(M_0) = 0$.*

THEOREM **2.8.** *$M_0$ is either empty, closed or a countable union of closed sets.*

*Proof.* Define $\Gamma_n = \{s \in \widetilde{M}$ : if $G$ open in $\widetilde{M}$ containing $s$, $\rho-$ diameter $< \frac{1}{n}$, then $K^*_{[G]}(s,x_0) \le \frac{1}{2}\}$. Take $s \in \Gamma_n$, for $m > n$, $H$ open in $\overline{M}$, $\rho-$ diameter $< \frac{1}{m}$ containing $s$. Then $H \subseteq B(s,\frac{1}{m}) \subseteq B(s,\frac{1}{n})$, $B$ open balls in $\overline{M}$.

Therefore $K^*_{[H]}(s,x_0) \le K^*_{[B(s,\frac{1}{m})]}(s,x_0) \le K^*_{[B(s,\frac{1}{n})]}(s,x_0) \le \frac{1}{2}$, which implies $s \in \Gamma_m$, hence, $\Gamma_n$ is an ascending sequence.

We claim that $\Gamma_n$ is closed. If true, let $s \in \Gamma_n$, $G$ open containing $s$ with diameter $< \frac{1}{n}$. Then, $\psi(s) = K_{\{s\}}(s,x_0) \le K^*_{[G]}(s,x_0) \le \frac{1}{2} < 1$, hence $\psi(s) = 0$ and $\Gamma_n \subseteq M_0$.

Assume that $s \in M_0$, denote $G_n = \{x \in \overline{M} : \rho(s,x) < \frac{1}{n}\}$. Then $\overline{G_n} \downarrow s$, $K^*_{[G_n]}(s,x_0) \downarrow K_{\{s\}}(s,x_0) = \psi(s) = 0$. Therefore $K^*_{[G_n]}(s,x_0) \le \frac{1}{2}$ for $n$ large enough and hence $s \in \Gamma_n$ for $n$ large enough. Hence, $M_0 = \cup_n \Gamma_n$ is either closed or a countable union of closed sets.

To prove that $\Gamma_n$ is closed, let $s_0$ be a limit point of $\Gamma_n$, $G$ open containing $s_0$ with $\rho$ diameter $< \frac{1}{n}$. Therefore, there exists $s_i \to s_0$ such that $s_i \in \Gamma_n$ for all $i$, i.e., $K^*_{[G]}(s_i, x_0) \leq \frac{1}{2}$, $i$ large enough. $K^*_{[G]}(s, x_0)$ lower semi-continuous, then,

$$K^*_{[G]}(s_0, x_0) \leq \varliminf_{i\to\infty} K^*_{[G]}(s_i, x_0) \leq \frac{1}{2}$$

Therefore, $s_0 \in \Gamma_n$. □

LEMMA **2.3.** $u_{\Gamma_n}(x) = 0$ *for* $u(x)$ *positive harmonic in* $M$.

*Proof.* Since $\Gamma_n$ is closed and compact, there exists finite cover $\{A_1, ..., A_k\}$ with diameter $< \frac{1}{n}$. Then, $u_{\Gamma_n} \leq \sum_{i=1}^k u_{A_i}(x)$. We claim that $u_{A_i}(x) = 0$ for all $i = 1, ..., k$. Take $A$ from $\Gamma_n$'s finite cover, $G$ open containing $A$ with diameter $< \frac{1}{n}$. Then, $K^*_{[G]}(s, x_0) \leq \frac{1}{2}$ for $s \in A$. Consider functions $v(x)$ of the form $v(x) = \int_A K(x, y) d\mu(y)$. Approximate $v(x)$ by $v_m(x) = \sum_{i=1}^m c_i K(s_i, x)$, $c_i > 0$, $s_i \in A$. Then,

$$\begin{aligned} (v_m)^*_{[G]}(x_0) &= \sum_{i=1}^m c_i K^*_{[G]}(s_i, x_0) \leq \frac{1}{2} \sum_{i=1}^m c_i \\ &= \frac{1}{2} \sum_{i=1}^m c_i K(s_i, x_0) = \frac{1}{2} v_m(x_0) \end{aligned}$$

Therefore, $v_A(x_0) \leq v^*_{[G]}(x_0) \leq \varliminf (v_m)^*_{[G]}(x_0) \leq \frac{1}{2} \lim v_m(x_0) = \frac{1}{2} v(x_0)$. Hence, $u_A(x_0) = (u_A)_A(x_0) \leq \frac{1}{2} u_A(x_0)$, i.e., $u_A(x_0) = 0$ and $u_A(x) = 0$. Therefore, $u_{\Gamma_n}(x) = 0$. □

LEMMA **2.4.** *Let* $u(x)$ *be positive harmonic in* $M$, $\varepsilon > 0$. *Then* $\exists$ *closed subsets* $A$ *of* $M_1$ *such that* $u(x_0) \leq u_A(x_0) + \varepsilon$.

*Proof.* Define $\Gamma_{m,n} = \{s \in \widetilde{M} : \rho(s, \Gamma_n \leq \frac{1}{m}\}$. Therefore, $\Gamma_{m,n} \downarrow \Gamma_n$ as $m \to \infty$. Since $u_{\Gamma_n}(x) = 0$, for $\varepsilon > 0$, choose $m_n$ large enough such that $u_{\Gamma_{m_n,n}}(x_0) < \frac{\varepsilon}{2^n}$. Define $C_n = \Gamma_{m_1,1} \cup \Gamma_{m_2,2} \cup ... \cup \Gamma_{m_n,n}$. Since, $\Gamma_n$ ascending and closed , $C_n$

ascending and closed. Define $A_n = \overline{\widetilde{M}\backslash C_n}$, $\rho(A_n, \Gamma_n) \geq \frac{1}{m_n}$. Then, $A_n$ descending, $\cap A_n = A, A \cap \Gamma_n = \phi$ for all $n$, i.e, $A \subset M_1$. $u_{C_n}(x_0) \leq \sum_{i=1}^{n} u_{\Gamma_{m_i,i}}(x_0) \leq \sum_{i=1}^{n} \frac{\varepsilon}{2^i} < \varepsilon$. Therefore,

$$u(x_0) = u_{\widetilde{M}}(x_0) = u_{A_n \cup C_n}(x_0) \leq u_{A_n}(x_0) + u_{C_n}(x_0) < u_{A_n}(x_0) + \varepsilon$$

Taking $n \to \infty$, $u(x_0) < u_A(x_0) + \varepsilon$. □

LEMMA **2.5.** *Let $A$, $B$ be closed subsets of $\widetilde{M}$, $A \cap B = \phi$, $B \subset M_1$, $\varepsilon > 0$. Then, there exists $G$ open containing $A$ such that for $s \in B$, $K^*_{[G]}(s, x_0) < \varepsilon$.*

*Proof.* Let $G_1$, $G_2$,... be descending open sets, containing $A$, $x_0 \in \overline{G_n}$. $\cap \overline{G_n} = A$, therefore, $x_0 \notin A$. If the hypothesis fails, for each $n$, there exists $s_n \in B$, $\delta > 0$ independent of $n$ such that $K^*_{[G]}(s_n, x_0) \geq \delta$. We have, $K^*_{[G_n]}(s_n, x) = \int_{\overline{G_n}} K(x, y) d\mu_n(y)$, $\mu_n$ a measure over $\overline{G_n}$ with total mass equals $K^*_{[G_n]}(s_n, x_0) \geq \delta$. Since $K^*_{[G_n]}(s_n, x_0) \leq K(s_n, x_0) = 1$, therefore, there exists subsequence of $\{\mu_n\}$ converges weakly to $\mu_0$ over $A$, total mass $\geq \delta$.
Since $B$ compact, $s_n \to s_0 \in B$. Therefore,

$$K(s_0, x) \leftarrow K(s_n, x) \geq K^*_{[G_n]}(s_n, x) = \int_{\overline{G_n}} K(x, y) d\mu_n(y) \to \int_A K(x, y) d\mu_0(y)$$

That is, $K(s_0, x) \geq \int_A K(x, y) d\mu_0(y) > 0$ (since $\mu_0(a) > 0$).
$K_A(s_0, x) > 0$, $s_0 \in B$ implies $K(s_0, x)$ minimal, previous corollary implies that $s_0 \in A$, which is a contradiction. □

LEMMA **2.6.** *Let $A \subseteq \widetilde{M}$ closed, $E \subseteq M_1$ Borel, $A \cap E = \phi$. Then, for $u(x)$ harmonic of the form*

$$u(x) = \int_E K(x, y) d\mu(y),$$

*we have $u_A(x) = 0$*

*Proof.* Consider for $B \subseteq M_1$, closed, $A \cap B = \phi$, $v_m(x) = \sum_{i=1}^{m} c_i K(s_i, x)$, $c_i > 0$, $s_i \in B$. Then, for $\varepsilon > 0$, $\exists G$ open containing $A$ such that $K^*_{[G]}(s_i, x_0) < \varepsilon$ for all

$s \in B$. Therefore,

$$(v_m)^*_{[G]}(x_0) = \sum_{i=1}^{m} c_i K^*_{[G]}(s, x_0) \leq \varepsilon \sum_{i=1}^{m} c_i = \varepsilon \sum_{i=1}^{m} c_i K(s_i, x_0) = \varepsilon v_m(x_0)$$

Then, approximate $u(x)$ by $v_m(x)$,

$$u_A(x_0) \leq u^*_{[G]}(x_0) \leq \underline{\lim}\,(v_m)^*_{[G]}(x_0) \leq \underline{\lim}\varepsilon v_m(x_0) = \varepsilon u(x_0)$$

$\varepsilon$ arbitrary, therefore, $u_A(x_0) = 0$ and $u_A(x) = 0$. For any $E \subseteq M_1$ Borel, write $E = B \cup C$, $B$ closed, $\mu(C) \leq \eta$, $\eta > 0$. Then,

$$u(x) = \int_B K(x,y)d\mu(y) + \int_C K(x,y)d\mu(y) = u_1(x) + u_2(x)$$

We have $(u_1)_A(x_0) = 0$. Since $\eta > 0$ arbitrary, $u_A(x_0) = 0$ and $u_A(x) = 0$. □

THEOREM **2.9.** *If $u(x)$ non-negative harmonic in $M$, then $u(x)$ admits exactly one canonical representation. Where $\mu$ is characterized by the relation*

$$u_A(x) = \int_A K(x,y)d\mu(y),$$

*$A$ closed in $\widetilde{M}$.*

*Proof.* Let $\varepsilon_m \downarrow 0$, then, $\exists A_1 \subseteq M_1$ depending on $\varepsilon_1$, $u(x)$ such that $u(x_0) - u_A(x_0) \leq \varepsilon_1$. Write $u(x) = u_A(x) + (u(x) - u_A(x)) = u_1(x) + u_1'(x)$. Then, $\exists \mu_1$ with total mass in $A_1$ canonical such that $u_1(x) = \int_{\widetilde{M}} K(x,y)d\mu_1(y)$ and $u_1'(x_0) \leq \varepsilon_1$. Next, split $u_1'(x) = u_2(x) + u_2'(x)$ such that $u_2(x) = \int_{\widetilde{M}} K(x,y)d\mu_2(y)$, $u_2'(x_0) < \varepsilon_2$ and $\mu_2$ canonical.
Therefore,

$$u(x) = \sum_{i=1}^{m} u_i(x) + u_m'(x), \;\; u_m'(x_0) < \varepsilon_m$$

Since $u_m'(x_0) \leq \varepsilon_m \to 0$, $u_m'(x)$ non-negative harmonic, we have $u_m'(x)$ tends to zero point-wisely as $m \to \infty$. Therefore, $u(x) = \sum_{i=1}^{\infty} u_i(x)$. Since any partial sum $\nu_k = \sum_{i=1}^{k} \mu_i$ represents $\sum_{i=1}^{k} u_i(x) \leq u(x)$ which is positive harmonic, therefore, $\nu_k$ have total mass lesser then or equal to $u(x_0)$. Hence, $\nu_k \rightharpoonup \mu$ passing to subsequence if necessary. That is, $\sum_{i=1}^{k} u_i(x) = \int_{\widetilde{M}} K(x,y)d\nu_k$. Take $k \to \infty$, $u(x) = \int_{\widetilde{M}} K(x,y)d\mu$, $\mu$ canonical.

We now want to show that $\mu$ is unique. Suppose $\mu$ canonical, representing $u(x)$. Define $u(E,x) = \int_E K(x,y)d\mu(y)$, $E$ Borel subset of $\widetilde{M} = M_1 \cup M_0$. $\mu$ canonical implies $u(E,x) = u(M_1 \cap E, x)$. Let $A \subset \widetilde{M}$ closed. Then,

$$u(x) = u(\widetilde{M}, x) = u(M_1, x) = u(M_1 \cap A, x) + u(M_1 \backslash A, x).$$

Therefore,

$$u_A(x) = u_A(M_1 \cap A, x) + u_A(M_1 \backslash A, x) = u_A(M_1 \cap A, x).$$

Now, let $A_n \subset \widetilde{M}$ closed such that $\rho(A,x) \le \frac{1}{n}$ for all $x \in A_n$, $B_n = \overline{\widetilde{M} \backslash A_n}$. $\therefore A \subset A_n$. Then, $A_n$, $B_n$ closed, $A_n \cup B_n = \widetilde{M}$, $A \cap B_n = \phi$.

$$\begin{aligned} u(M_1 \cap A, x) &= u_{\widetilde{M}}(M_1 \cap A, x) = u_{A_n \cap B_n}(M_1 \cap A, x) \\ &= u_{A_n}(M_1 \cap A, x) + u_{B_n}(M_1 \cap A, x) = u_{A_n}(M_1 \cap A, x) \\ &\le u(M_1 \cap A, x) \end{aligned}$$

Hence $u_{A_n}(M_1 \cap A, x) = u(M_1 \cap A, x)$. Take $n \to \infty$, $A_n \downarrow A$, $u_A(M_1 \cap A, x) = u(M_1 \cap A, x)$. Therefore, $u_A(x) = u_A(M_1 \cap A, x) = u(M_1 \cap A, x) = u(A,x) = \int_A K(x,y)d\mu(y)$, $A$ closed in $\widetilde{M}$. Therefore, $\mu(A) = u_A(x_0)$, hence unique. Then, since $u(x) = u_{\widetilde{M}}(x)$, we got the representation formula. $\square$

# Chapter 3

# The Geometric boundary and the Martin boundary

In this chapter, we will consider the relation between the Martin boundary and geometric boundary of a simply connected complete manfiold $M$ with sectional curvatures satisfying $-b^2 \leq K_M \leq -a^2 < 0$ for some positive constants $b > a > 0$. We will give an exposition of the result of Anderson and Schoen [A-S].

In the following sections, if no ambiguity occurs, we always denote $C_0(\theta)$ as $C(\theta)$ and $T_0(\theta, R)$ as $T(\theta, R)$, where $0 \in M$. We will first give some preliminary results that are required to prove the main theorem.

## 3.1 Estimates for harmonic functions in cones

Lemma **3.1.** *Suppose that $\varphi$ is a Lipschitz function on $M(\infty)$ with a radial extension to $M$, denote*

$$\overline{\varphi}(x) = \frac{\int_M \chi(\rho^2(x,y))\varphi(y)dy}{\int_M \chi(\rho^2(x,y))},$$

*where $\chi : \mathbb{R} \to \mathbb{R}$ is a fixed $C^2$ approximation to $\chi_{[0,1]}$. Then, we have $\|H\overline{\varphi}(x_0)\| \leq Ce^{-ad}$, $\|\nabla\overline{\varphi}(x_0)\| \leq Ce^{-ad}$, where $d = \rho(0, x_0)$.*

*Proof.* Denote $\overline{\chi}(\rho) = \chi(\rho^2)$, then we have $H(\chi(\rho^2)) = \overline{\chi}'' d\rho \otimes d\rho + \overline{\chi}' H(\rho)$. By Rauch comparison theorem, we have

$$a \coth a\rho(g - d\rho \otimes d\rho) \leq H(\rho) \leq b \coth b\rho(g - d\rho \otimes d\rho).$$

Then,

$$\begin{aligned} H(\chi(\rho^2)) &= \overline{\chi}'' d\rho \otimes d\rho + \overline{\chi}' H(\rho) \\ &\leq \overline{\chi}'' d\rho \otimes d\rho + \overline{\chi}' a \coth a\rho(g - d\rho \otimes d\rho) \\ &= (\overline{\chi}' a \coth a\rho)\, g + (\overline{\chi}'' - \overline{\chi}' a \coth a\rho)\, d\rho \otimes d\rho \\ &\leq k_1 g + k_2(g - h_{\alpha\beta} d\xi_\alpha \otimes d\xi_\beta) \leq kg. \end{aligned}$$

Also,

$$\begin{aligned} H(\overline{\varphi})(x_0) &= H(\overline{\varphi} - \varphi(x_0))(x_0) \\ &= H\left(\frac{\int_M \chi(\rho^2(x_0,y))(\varphi(y) - \varphi(x_0))dy}{\int_M \chi(\rho^2(x_0,y))dy}\right) \\ &= \int_M H\left(\frac{\chi(\rho^2(x_0,y))}{\int_M \chi(\rho^2(x_0,y))dy}\right)(\varphi(y) - \varphi(x_0))dy \\ &\leq \sup_{B(x_0,2)} |\varphi(y) - \varphi(x_0)| \int_M H\left(\frac{\chi(\rho^2(x_0,y))}{\int_M \chi(\rho^2(x_0,y))dy}\right) dy. \end{aligned}$$

Now, denote $u = \chi(\rho^2(x_0,y))$, $v = \int_M \chi(\rho^2(x_0,y))dy$. Then,

$$H\left(\frac{u}{v}\right) = \frac{1}{v^2}(vHu - 2du \otimes dv - uHv) + \frac{2u}{v^3} dv \otimes dv.$$

We have $du \otimes dv = \left(\overline{\chi}' \int_M \overline{\chi}' dy\right) d\rho \otimes d\rho \leq k_1 g$, $dv \otimes dv = \left(\int_M \overline{\chi}' dy\right)^2 d\rho \otimes d\rho \leq k_2 g$, $u$ bounded from above, $v$ bounded from below away from zero, $H(u) = H\chi(\rho^2)(x_0)$ bounded by $k_3 g$, $H(v) = H(\int_M \chi(\rho^2(x_0,y))dy) = \int_M H\chi(\rho^2)dy \leq k_4 g$. Therefore,

$$\begin{aligned} \|H\overline{\varphi}(x_0)\| &\leq C_1 \left\|H\left(\frac{u}{v}\right)(x_0)\right\| \cdot \sup_{B(x_0,2)} |\varphi(y) - \varphi(x_0)| \\ &\leq C_2 \sup_{B(x_0,2)} |\varphi(y) - \varphi(x_0)| \end{aligned}$$

Let $d$ be the distance between 0 and $x_0$, $\omega(d)$ be the maximum angle subtended at 0 by $B(x_0, 2)$. Take a point $x_1$ at $\partial B(x_0, 2)$ and denote its distance to 0 by $d_1$, $\overline{\theta}_1$ be the angle subtended by $x_0$ and $x_1$ at 0, $\theta_1$ be the corresponding angle in $\mathbb{H}(-a^2)$. By Toponogov comparison theorem, we have $\overline{\theta_1} \leq \theta_1$. Consider the cosine law on the hyperbolic space, we have

$$\cos\theta_1 = \coth ad_1 \coth ad - \frac{\cosh 2a}{\sinh ad_1 \sinh ad}.$$

Hence,

$$\begin{aligned} 1 - \frac{\theta_1^2}{2!} + \frac{\theta_1^4}{4!} + \dots &= \left(\frac{1+e^{-2ad_1}}{1-e^{-2ad_1}}\right)\left(\frac{1+e^{-2ad}}{1-e^{-2ad}}\right) - \frac{4\cosh 2a}{(e^{ad_1}-e^{-ad_1})(e^{ad}-e^{-ad})} \\ &\geq 1 - \frac{4\cosh 2a}{(e^{ad_1}-e^{-ad_1})(e^{ad}-e^{-ad})} \end{aligned}$$

$$\text{Hence, } \frac{\theta_1^2}{2!} - \frac{\theta_1^4}{4!} + \dots \leq \frac{4\cosh 2a}{(e^{ad_1}-e^{-ad_1})(e^{ad}-e^{-ad})}$$

Therefore,

$$\begin{aligned} \theta_1^2 &\leq \frac{C_3}{(e^{ad_1}-e^{-ad_1})(e^{ad}-e^{-ad})} \\ &= \frac{C_3 e^{-a(d+d_1)}}{(1-e^{-2ad_1})(1-e^{-2ad})} \leq C_4 e^{-a(d+d_1)} \leq C_5 e^{-2ad} \end{aligned}$$

Therefore, $\omega(d) \leq C_6 e^{-ad}$. Now, since $\varphi$ is radial outside a compact set, $|\varphi(y) - \varphi(x_0)| \leq C_7 \angle(x_0, y) \leq C_7\omega(d) \leq C_8 e^{-ad}$. Hence,

$$\|H\overline{\varphi}(x_0)\| \leq C \sup_{B(x_0,2)} |\varphi(y) - \varphi(x_0)| \leq C_0 e^{-ad},$$

and similarly for $\|\nabla\overline{\varphi}(x_0)\|$. $\square$

THEOREM **3.1.** *Let $\theta_0 \in (0, \pi)$, $h$ a positive harmonic function on $C(\theta_0)$, which is continuous in the closure of $C(\theta_0)$ and vanishes on $\overline{C(\theta_0)} \cap M(\infty)$. Then, we have*

$$\sup_{T(\frac{\theta_0}{2}, R_0)} h \leq C_1 \sup_{(\partial B_0(R_0)) \cap C(\frac{7}{8}, \theta_0)} h$$

$R_0 = C_2(\log\theta_0^{-1}) + C_3$.

*Proof.* First, we will construct a superharmonic function which dominates $h$ on the boundary of a cone. Suppose $\lambda > 0$ is a smooth function in $C(\theta_0)$ with values in $[0,1]$. Then,

$$\begin{aligned}\Delta h^\lambda &= h^\lambda\{|\nabla\lambda|^2(\log h)^2 + 2 < \nabla \log h, \nabla\lambda > + 2\lambda \log h < \nabla \log h, \nabla\lambda > \\ &\quad + (\nabla\lambda)\log h - \lambda(1-\lambda)|\nabla \log h|^2\}\end{aligned}$$

Next, consider for $x \in \partial C(\frac{7}{8}\theta_0)$, $\rho(x) \geq C_4 \log\theta_0^{-1} + C_5$ large enough, we have $dist(x, \partial C(\theta_0) \cup C(\frac{3}{4}\theta_0)) \geq 2$. Then, for $x \in C(\frac{7}{8}\theta_0)$, the interior Harnack inequality implies that

$$\sup_{B(x,1)} h \leq \tilde{C} \inf_{B(x,1)} h$$

Let $R_0 = C_4 \log\theta_0^{-1} + C_5$. Suppose $h \leq 1$ on $\partial B_0(R_0) \cap C(\frac{7}{8}\theta_0)$. Pick $x_0 \in \partial B_0(R_0) \cap C(\frac{7}{8}\theta_0)$, $x_1 \in \partial B(x_0, 1)$. Then, $h(x_2) \leq \tilde{C}h(x_1) \leq \tilde{C}^2$. Furthermore, choose $x_2 \in \partial B(x_1, 1)$. Then, $h(x_2) \leq \tilde{C}h(x_1) \leq \tilde{C}^2$. Therefore, if we choose $x_n \in \gamma_{x_0}(\frac{7}{8}\theta_0)$ such that $\rho(x_n) = n + \rho(x_0)$, $h(x_n) \leq \tilde{C}^n$.
Now, choose $x_\xi \in \gamma_{x_0}(\frac{7}{8}\theta_0)$ such that $\rho(x_\xi) = n + \xi + \rho(x_0)$, $0 < \xi < 1$. We have $h(x_\xi) \leq \tilde{C}h(x_n)$. Hence $\frac{1}{\tilde{C}}h(x_\xi) \leq h(x_n) \leq \tilde{C}^n \leq \tilde{C}^{n+\xi}$,

$$h(x_\xi) \leq \tilde{C}\tilde{C}^{n+\xi} = \tilde{C}e^{(n+\xi)\log\tilde{C}} = \tilde{C}_2 e^{\tilde{C}_3\rho(x_x i)} \leq e^{C_6\rho(x_\xi)}$$

for $x \in \gamma_{x_0}(\frac{7}{8}\theta_0)$.
Now, let $\lambda_1$ be a Lipschitz function of $\theta$ with

$$\lambda_1(\theta) = \begin{cases} \frac{1}{2} \ , & \theta \in [0, \frac{5}{8}\theta_0] \\ 1 \ , & \theta \in [\frac{6}{8}\theta_0, \frac{7}{8}\theta_0] \end{cases}$$

$$|\lambda_1'(\theta)| \leq C_7\theta_0^{-1}$$

Let $\chi$ be a smooth approximation of $\chi_{[0,\frac{1}{8}\theta_0]}$. Define $\lambda$ on $T(\frac{7}{8}\theta_0, R_0)$ by $\lambda = \dfrac{\int_M \chi(\rho^2)\lambda_1}{\int_M \chi(\rho^2)}$.

We have $|\nabla\lambda| + |H\lambda| \le C_8 e^{-C_9\rho(x)}$, $\sup_{B(1)} |\nabla \log h| \le C$, $\log h \le C_6\rho$, therefore,

$$\begin{aligned}
\Delta h^\lambda &= h^\lambda\{|\nabla\lambda|^2(\log h)^2 + 2 < \nabla \log h, \nabla\lambda > + 2\lambda \log h < \nabla \log h, \nabla \lambda > \\
&\quad +(\nabla\lambda)\log h - \lambda(1-\lambda)|\nabla \log h|^2\} \\
&\le h^\lambda\{|\nabla\lambda|^2(\log h)^2 + 2\log h|\nabla\lambda|\cdot|\nabla\log h| + |\nabla\lambda|\log h + 2|\nabla\lambda|\cdot|\nabla \log h|\} \\
&\le h^\lambda\{C_8^2 e^{-C_9\rho}C_6^2\rho + 2C_6\rho C_8 e^{-C_9\rho}C + C_8 e^{-C_9\rho}C_6\rho + 2C_8 e^{-C_9\rho}C\} \\
&\le C_{10}e^{-C_{11}\rho}h^\lambda
\end{aligned}$$

Now, choose $\varepsilon_0 \in (0, \frac{1}{4})$ such that $h^{\varepsilon_0(x)} \le e^{\varepsilon_0 C_6\rho} \le e^{\frac{1}{2}C_{11}\rho}$ in $T(\frac{7}{8}\theta_0, R_0)$. Then,

$$\begin{aligned}
\Delta h^\lambda &\le C_{10}e^{-C_{11}\rho}h^\lambda \\
&= C_{10}e^{-\frac{1}{2}C_{11}\rho}h^\lambda \cdot e^{-\frac{1}{2}C_{11}\rho} \\
&\le C_{10}e^{-\frac{1}{2}C_{11}\rho}h^{\lambda-\varepsilon_0} = C_{10}e^{-C_{12}\rho}h^{\lambda-\varepsilon_0}
\end{aligned}$$

Consider $\dfrac{1+h^{1-\frac{\varepsilon_0}{2}}}{h^{\lambda-\varepsilon_0}} = h^{\varepsilon_0-\lambda} + h^{1+\frac{\varepsilon_0}{2}-\lambda}$, $\varepsilon_0 - \lambda < 0$, $1 + \frac{\varepsilon_0}{2} - \lambda > 0$, there exists $k_1$ such that $1 + h^{1-\frac{\varepsilon_0}{2}} \ge k_1 h^{\lambda-\varepsilon_0}$. Therefore $\Delta h^\lambda \le C_{13}e^{-C_12\rho}(1+h^{\alpha_0})$, $\alpha_0 = 1 - \frac{\varepsilon_0}{2}$. For $\delta > 0$, consider

$$\begin{aligned}
&\Delta(e^{-\delta\rho}h^{\alpha_0}) \\
&= h^{\alpha_0}\Delta e^{-\delta\rho} + e^{-\delta\rho}\Delta h^{\alpha_0} + 2 < \nabla e^{-\delta\rho}, \nabla h^{\alpha_0} > \\
&\le h^{\alpha_0}e^{-\delta\rho}(\delta^2 - \delta\Delta\rho) + 2\delta e^{-\delta\rho}\alpha_0 h^{\alpha_0-1}|\nabla h| + e^{-\delta\rho}\Delta h^{\alpha_0} \\
&= -h^{\alpha_0}\delta e^{-\delta\rho}(\Delta\rho - \delta) + 2\delta e^{-\delta\rho}\alpha_0 h^{\alpha_0-1}|\nabla h| + e^{-\delta\rho}(-\alpha_0(1-\alpha_0)|\nabla \log h|^2)h^{\alpha_0} \\
&\le -h^{\alpha_0}\delta e^{-\delta\rho}(C_{14} - \delta) + 2\delta e^{-\delta\rho}h^{\alpha_0-1}|\nabla h| - \alpha_0(1-\alpha_0)e^{-\delta\rho}h^{\alpha_0}\frac{1}{h^2}|\nabla h|^2 \\
&= -h^{\alpha_0}\delta e^{-\delta\rho}(C_{14} - \delta) + 2\left(\delta e^{-\frac{\delta\rho}{2}}h^{\frac{\alpha_0}{2}}k_2\right)\left(\frac{1}{k_2}e^{-\frac{\delta\rho}{2}}h^{\frac{\alpha_0}{2}-1}|\nabla h|\right) \\
&\quad -\alpha_0(1-\alpha_0)e^{-\delta\rho}h^{\alpha_0-2}|\nabla h|^2 \\
&\le -h^{\alpha_0}\delta e^{-\delta\rho}(C_{14} - \delta) + (\delta^2 e^{-\delta\rho}h^{\alpha_0}k_2^2) + \left(\frac{1}{k_2^2}e^{-\delta\rho}h^{\alpha_0-2}|\nabla h|^2\right) \\
&\quad -\alpha_0(1-\alpha_0)e^{-\delta\rho}h^{\alpha_0-2}|\nabla h|^2 \\
&= -h^{\alpha_0}\delta e^{-\delta\rho}(C_{14} - \delta) + C_{15}\delta^2 e^{-\delta\rho}h^{\alpha_0} + \left(\frac{1}{k_2^2} - \alpha_0(1-\alpha_0)\right)e^{-\delta\rho}h^{\alpha_0-2}|\nabla h|^2
\end{aligned}$$

Therefore, we choose $k_2$ such that $\frac{1}{k_2^2} - \alpha_0(1-\alpha_0) < 0$. Then,

$$\begin{aligned}\Delta(e^{-\delta\rho}h^{\alpha_0}) &\leq -h^{\alpha_0}\delta e^{-\delta\rho}(C_{14}-\delta) + C_{15}\delta^2 e^{-\delta\rho}h^{\alpha_0} \\ &= \delta h^{\alpha_0}e^{-\delta\rho}(C_{14}-\delta-C_{19}\delta) \\ &= -C_{16}\delta h^{\alpha_0}e^{-\delta\rho}\end{aligned}$$

$$\text{Hence, we have } \begin{cases}\Delta(e^{-\delta\rho h^{\alpha_0}}) &\leq -C_{16}\delta h^{\alpha_0}e^{-\delta\rho} \\ \Delta e^{-\delta\rho} &\leq -C_{16}\delta e^{-\delta\rho}\end{cases}$$

Next, we consider

$$\begin{aligned}&\Delta(h^\lambda + C_{17}e^{-\delta\rho}(h^{\alpha_0}+1)) \\ =\ &\Delta h^\lambda + C_{17}\Delta(e^{-\delta\rho}h^{\alpha_0}) + C_{17}e^{-\delta\rho} \\ \leq\ &C_{13}e^{-C_{12}\rho}(h^{\alpha_0}+1) - C_{16}C_{17}\delta e^{-\delta\rho}h^{\alpha_0} - C_{16}C_{17}\delta e^{-\delta\rho} \\ \leq\ &C_{13}e^{-\delta\rho}(h^{\alpha_0}+1) - C_{16}C_{17}\delta e^{-\delta\rho}h^{\alpha_0} - C_{16}C_{17}\delta e^{-\delta\rho} \\ =\ &(C_{13}-C_{16}C_{17}\delta)e^{-\delta\rho}h^{\alpha_0} + (C_{13}-C_{16}C_{17}\delta)e^{-\delta\rho} \leq 0\end{aligned}$$

choosing $\delta < C_{12}$, $C_{17}$ large enough. Therefore, $\Delta(h^\lambda + C_{17}e^{-\delta\rho}(e^{\alpha_0}+1)) \leq 0$. Denote $F = h^\lambda + C_{17}e^{-\delta\rho}(h^{\alpha_0}+1) \geq h$. Note that $h \leq 1$ on $C(\frac{7}{8}\theta_0) \cap \partial B_0(R_0)$, then, $F = h^\lambda + C_{17}e^{-\delta\rho}(h^{\alpha_0}+1) \geq h + C_{17}e^{-\delta\rho}(h^{\alpha_0}+1) \geq h$. Therefore, $F \geq h$ on $\partial T(\frac{7}{8}\theta_0, R_0)$, and hence on $T(\frac{7}{8}\theta_0, R_0)$. This implies $h \leq h^\lambda + C_{18}h^{\alpha_0} + C_{18}$ on $T(\frac{7}{8}\theta_0, R_0)$.

Consider on $T(\frac{1}{2}\theta_0, R_0)$, $\lambda \equiv \frac{1}{2}$, $h^\lambda = h^{\frac{1}{2}} \leq \frac{1}{4}h + 1$. For some constant $C_{19}$, we have $C_{18}h^{\alpha_0} \leq \frac{1}{4}h + C_{19}$, hence $h \leq h^\lambda + C_{18}h^{\alpha_0} + C_{18} \leq \frac{1}{4}h + 1 + \frac{1}{4}h + C_{19} + C_{18}$, i.e., $h \leq C_{20}$ on $T(\frac{1}{2}\theta_0, R_0)$. Therefore,

$$\sup_{T(\frac{1}{2}\theta_0, R_0)} h \leq C_{20}$$

Finally, we consider the general case for $h$. Let $M(R_0) = \sup_{C(\frac{7}{8}\theta_0, R_0)\cap\partial B_0(R_0)} h$. We have,

$$\sup_{T(\frac{1}{2}\theta_0, R_0)} \frac{h}{M(R_0)} \leq C_{20}$$

$$\sup_{T(\frac{1}{2}\theta_0,R_0)} h \leq C_{20} \sup_{C(\frac{7}{8}\theta_0,R_0)\cap\partial B_0(R_0)} h$$

□

Corollary **3.1.** *If $h$ is a positive harmonic function defined on $C(\frac{\pi}{4})$, vanishing continuously on $C(\frac{\pi}{4}) \cap M(\infty)$, then,*

$$h(x) \leq C_1 e^{-C_2\rho} h(0')$$

*for $x \in T(\frac{\pi}{8},1)$, $0' = exp_0 v_0$, where $v_0$ is the unit tangent in the direction of the cone at* 0.

*Proof.* Using Theorem 3.1, we have for $\frac{3\pi}{16} < \alpha < \frac{\pi}{4}$,

$$\sup_{T(\frac{3\pi}{16},R_0)} \leq C_1 \sup_{\partial B_0(R_0)\cap C(\alpha)} .h$$

Then, using Interior Harnack Inequality, $\sup_{T(\frac{3\pi}{16},1)} h \leq C_2 h(0')$. Let $\varphi$ be a Lipschitz function of angle $\theta$ such that

$$\varphi(\theta) = \begin{cases} 0 & , \quad \theta \leq \frac{\pi}{8} + \varepsilon \\ 1 & , \quad \theta \geq \frac{3\pi}{16} - \varepsilon \end{cases}$$

$|\nabla\varphi| \leq C_3$ in $S_0(1)$. Define $g = C_4 e^{-\delta\rho} + \overline{\varphi}$, where $\overline{\varphi}$ is the average of $\varphi$, $\overline{\varphi}(\frac{\pi}{8}) \equiv 0$, $\overline{\varphi}(\frac{3\pi}{16}) \equiv 1$. Then,

$$\begin{aligned} \Delta g &= C_4 \Delta e^{-\delta\rho} + \Delta\varphi = C_4 \delta e^{-\delta\rho}(\delta - \Delta\rho) + \Delta\overline{\varphi} \\ &\leq C_4 \delta e^{-\delta\rho}(\delta - a(n-1)) + C_5 e^{-ap} \\ &\leq \left(C_4\delta(\delta - a(n-1)) + C_5\right) e^{-\delta\rho} \end{aligned}$$

choosing $\delta < a$. Also, by choosing $C_4$ large enough, $\delta$ small enough such that $\delta - a(n-1) < 0$, $C_4\delta(\delta - a(n-1)) + C_5 < 0$, we have $\Delta g \leq 0$ on $T(\frac{3\pi}{16},1)$. Since $\overline{\varphi} \equiv 1$ on $\theta = \frac{3\pi}{16}$, we have $g \geq 1$ on $\partial T(\frac{3\pi}{16},1)$. Therefore, $\frac{h(x)}{h(0')} \leq C_6 g(x)$ on $\partial T(\frac{3\pi}{16},1)$ and heence on $T(\frac{3\pi}{16},1)$, since $h(x)$ harmonic and $g(x)$ superharmonic. $\overline{\varphi} \equiv 0$ on $T(\frac{\pi}{8},1) \implies h(x) \leq C_6 C_4 e^{-\delta\rho} h(0') = C_7 e^{-C_8\rho} h(0')$. □

## 3.2 A Harnack Inequality at Infinity

Theorem 3.2 (**Comparison Principle**). *Let $u$, $v$ be positive harmonic functions defined in $C(\frac{\pi}{4})$, continuous up to $\partial C(\frac{\pi}{4})$, vanishing on $\overline{C(\frac{\pi}{4})}\cap M(\infty)$. Then, there exists $C_1 > 0$ constant such that*

$$\frac{1}{C_1}\frac{u(0')}{v(0')} \leq \frac{u(x)}{v(x)} \leq C_1\frac{u(0')}{v(0')}$$

*$x \in T(\frac{\pi}{8}, 1)$, $0' = exp_0 v_0$.*

*Proof.* We can assume that $u(0') = v(0') = 1$. Therefore, we have to prove that $\frac{1}{C_1} \leq \frac{u(x)}{v(x)} \leq C_1$. Similar to Corollary 3.1, we can prove that $C_4 e^{-C_5\rho} \leq u(x)$. Hence,

$$C_4 e^{-C_5\rho} \leq u(x) \leq C_2 e^{-C_3\rho}, \qquad x \in T(\frac{3\pi}{16}, 1)$$

Denote $\rho_1(x) = \log\left(u(x)^{-1}\right)$. Then,

(i) $C_3\rho - C_6 \leq \rho_1 \leq C_5\rho + C_6$

(ii) $|\nabla\rho_1| \leq C_6$, $\Delta\rho_1 = \Delta\log u^{-1} = -\left(\frac{1}{u}\Delta u - \frac{1}{u^2}|\nabla u|^2\right) = |\nabla\log u|^2 = |\nabla\rho_1|^2$

(iii) $\Delta\rho_1 = |\nabla\rho_1|^2$.

Consider $F = u^\lambda \cdot \psi(\rho_1 - e^{-\alpha\rho})$, $\lambda$ a function with values in $[0,1]$, $\psi(t)$ a bounded function, $\alpha$ a constant such that $2t^{-2} \geq \psi(t) \geq t^{-2}$, $\psi''(t) \leq 0$, $\psi(t) \geq 1$ for $t \geq 1$. Then,

$$\begin{aligned}
\Delta\psi &= \psi' \cdot (\Delta\rho_1 - \Delta e^{-\alpha\rho}) = \psi' \cdot (|\nabla\rho_1|^2 - \alpha e^{-\alpha\rho}(\alpha - \Delta\rho)) \\
&\leq \psi' \cdot \left(|\nabla\log u|^2 - \alpha e^{-\alpha\rho}(\alpha - b(n-1))\right) \\
&\leq \psi' \cdot \left(|\nabla\log u|^2 - \frac{\alpha^2}{2}e^{-\alpha\rho}\right)
\end{aligned}$$

Where we choose $\alpha$ such that $\alpha - b(n-1) \geq \frac{\alpha}{2}$.

$$\begin{aligned}
2 < \nabla\psi, \nabla u^\lambda > &= 2\psi' u^\lambda < \nabla\rho_1 - \nabla e^{-\alpha\rho}, \log u \nabla\lambda + \lambda \log u > \\
&= 2\psi' u^\lambda < -\nabla \log u + \alpha e^{-\alpha\rho}\nabla\rho, \log u\nabla\lambda + \lambda\log u > \\
&= 2\psi' u^\lambda(-\lambda|\nabla \log u|^2 + \lambda\alpha e^{-\alpha\rho} < \nabla\rho, \nabla\log u > \\
&\quad - \log u < \nabla\lambda, \nabla\log u > + \alpha e^{-\alpha\rho}\log u < \nabla\rho, \nabla\lambda >) \\
&\leq 2\psi' u^\lambda(-\lambda|\nabla\log u|^2 + \alpha e^{-\alpha\rho}|\nabla\log u| \\
&\quad + |\log u|\cdot|\nabla\lambda|\cdot|\nabla\log u| + |\log u|\alpha e^{-\alpha\rho}|\nabla\lambda|) \\
&\leq 2\psi' u^\lambda(-\lambda|\nabla\log u|^2 + C_8\alpha e^{-\alpha\rho} + C_8\rho|\nabla\lambda|\cdot|\nabla\log u| \\
&\quad + C_8\alpha\rho e^{-\alpha\rho}|\nabla\lambda|)
\end{aligned}$$

Then,

$$\begin{aligned}
\Delta F &= \Delta\left(u^\lambda\cdot\psi(\rho_1 - e^{-\alpha\rho})\right) = u^\lambda\Delta\psi + 2 < \Delta u^\lambda, \Delta\psi > + \psi\Delta u^\lambda \\
&\leq \psi\Delta u^\lambda + \psi' u^\lambda(|\nabla\log u|^2 - \frac{\alpha^2}{2}e^{-\alpha\rho} - 2\lambda|\nabla\log u|^2 + 2C_8\alpha e^{-\alpha\rho} \\
&\quad + 2C_8\rho|\nabla\lambda|\cdot|\nabla\log u| + 2C_8\alpha\rho e^{-\alpha\rho}|\nabla\lambda|) \\
&= \psi\Delta u^\lambda + \psi' u^\lambda\{(1-2\lambda)|\nabla\log u|^2 + 2C_8\rho|\nabla\lambda|(|\nabla\log u| + \alpha e^{-\alpha\rho}) \\
&\quad - C_9\alpha^2 e^{-\alpha\rho}\}
\end{aligned}$$

By supposing $|\nabla\lambda| \leq C_{10}e^{-C_{11}\rho}$, fix $\alpha$, we get,

$$2C_8\alpha\rho e^{-\alpha\rho}|\nabla\lambda| \geq C_8'\rho e^{-(\alpha+C_{11})\rho} \leq C_8' e^{-C_8''\rho}, C_8'' > \alpha$$

Then, $C_8' e^{-C_8''\rho} - C_9' e^{-\alpha\rho} \leq -C_{12}e^{-\alpha\rho}$, $\rho > R_0$ large enough.

$$\Longrightarrow \Delta F \leq \psi\Delta u^\lambda + \psi' u^\lambda\left((1-2\lambda)|\nabla\log u|^2 - C_{12}e^{-\alpha\rho} + 2C_9\rho|\nabla\lambda|\cdot|\nabla\log u|\right)$$

Let $\varphi$ be a Lipschitz function of $\theta$ such that

$$\varphi(\theta) = \begin{cases} \varepsilon_0 & ,\theta \geq \frac{10\pi}{64} \\ 1 & ,\theta \leq \frac{9\pi}{64} \end{cases}$$

$|\varphi'(\theta)| \le C_{13}$, $\varepsilon_0 > 0$ small enough. For $\varepsilon_1 > 0$, define $\lambda_1 = 1 - (1-\overline{\varphi})^{\frac{2}{\varepsilon_1}}$, $\lambda_2 = 1 - (1-\overline{\varphi})^{\frac{1}{\varepsilon_1}}$ on $M$. Then, we have

$$\begin{cases} |\nabla\lambda_1| + |H\lambda_1| \le C_{14}e^{-C_{15}\rho}(1-\lambda_1)^{1-\frac{\varepsilon_1}{2}} \\ |\nabla\lambda_2| + |H\lambda_2| \le C_{14}e^{-C_{15}\rho}(1-\lambda_2)^{1-\varepsilon_1}, \mathrm{on} T(\frac{3\pi}{16}, R_0) \end{cases}$$

Define $\lambda = \lambda_1 - (1-\lambda_2)e^{-\delta_0\rho}$, $\delta_0 > 0$ constant. $\varphi(\theta) = \varepsilon_0$ on $\partial C(\frac{3\pi}{16}\backslash B_0(R_0)$. Then, for $R_0$ large enough, choose $\varepsilon_0$, $\varepsilon_1$ such that $\frac{1}{2}(1-(1-\varepsilon_0)^{\frac{2}{\varepsilon_1}}) \le \lambda \le 1-(1-\varepsilon_0)^{\frac{2}{\varepsilon_1}}$ By previous results, we have $v(x) \le C_e^{-C\rho}$ on $T(\frac{3\pi}{16}, 1)$. Choose $\varepsilon_1$, $\varepsilon_0$ such that $v \le u^\lambda$ on $\partial C(\frac{3\pi}{16})\backslash B_0(R_0)$.

$$\begin{aligned} \Delta u^\lambda &= u^\lambda(|\nabla\lambda|^2(\log u)^2 + 2\lambda\log u < \nabla\lambda, \log u > + \log u\Delta\lambda + 2 < \nabla\lambda, \nabla\log u > \\ &\quad -\lambda(1-\lambda)|\nabla\log u|^2) \\ &\le u^\lambda(C_{16}\rho^2|\nabla\lambda|^2 + C_{16}\rho|\nabla\lambda|\cdot|\nabla\log u| + \log u\Delta\lambda - \lambda(1-\lambda)|\nabla\log u|^2) \end{aligned}$$

Then,

$$\begin{aligned} \Delta F &\le \psi' u^\lambda\left((1-2\lambda)|\nabla\log u|^2 - C_{12}e^{-\alpha\rho} + 2C_8\rho|\nabla\lambda|\cdot|\nabla\log u|\right) \\ &\quad +\psi u^\lambda\left(C_{16}\rho^2|\nabla\lambda|^2 + C_{16}\rho|\nabla\lambda|\cdot|\nabla\log u| + \log u\nabla\lambda - \lambda(1-\lambda)|\nabla\log u|^2\right) \end{aligned}$$

Since $2t^{-2} \ge \psi'(t) \ge t^{-2} \implies \psi'(t)$ bounded. Therefore, $\exists$ constant $C_{17}$ such that $2C_8\psi' + C_{16}\psi \le C_{17}\psi$. Therefore,

$$\begin{aligned} \Delta F &\le \psi' u^\lambda\left((1-2\lambda)|\nabla\log u|^2 - C_{12}e^{-\alpha\rho}\right) \\ &\quad +\psi u^\lambda\left(C_{16}\rho^2|\nabla\lambda|^2 + C_{17}\rho|\nabla\lambda|\cdot|\nabla\log u| + \log u\Delta\lambda - \lambda(1-\lambda)|\nabla\log u|^2\right) \end{aligned}$$

Consider when $\lambda \ge \frac{3}{4}$,

$$\begin{aligned} C_{17}\rho\psi|\nabla\lambda|\cdot|\nabla\log u| &= \left((2\lambda-1)^{\frac{1}{2}}\rho^{-1}|\nabla\log u|\right)\left(C_{17}(2\lambda-1)^{-\frac{1}{2}}\rho^2\psi|\nabla\lambda|\right) \\ &\le (2\lambda-1)\rho^{-2}|\nabla\log u|^2 + \frac{1}{4}C_{17}(2\lambda-1)^{-1}\psi\rho^4\psi|\nabla\lambda|^2 \\ &\le (2\lambda-1)\psi'|\nabla\log u|^2 + C_{18}\rho^4\psi|\nabla\lambda|^2 \end{aligned}$$

Then, $$\begin{aligned} \Delta F &\le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + \psi u^\lambda\left(C_{16}\rho^2|\nabla\lambda|^2 + C_{18}\rho^4\psi|\nabla\lambda|^2 + (\log u)\Delta\lambda\right) \\ &\le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + \psi u^\lambda\left(C_{21}\varepsilon_0^{-1}\rho^4|\nabla\lambda|^2 + (\log u)\Delta\lambda\right) \end{aligned}$$

When $\frac{1}{2} \leq \lambda \leq \frac{3}{4}$, $\exists C$ such that $\lambda(1-\lambda) \geq C\varepsilon_0$. Therefore,

$$\begin{aligned} C_{17}\rho|\nabla\lambda| \cdot |\nabla \log u| &= \left(\sqrt{C\varepsilon_0}|\nabla \log u|\right)\left(\frac{\rho}{\sqrt{C\varepsilon_0}}C_{17}|\nabla\lambda|\right) \\ &\leq C\varepsilon_0|\nabla \log u|^2 + C_{20}\varepsilon_0^{-1}\rho^2|\nabla\lambda|^2 \\ &\leq \lambda(1-\lambda)|\nabla \log u|^2 + C_{20}\varepsilon_0^{-1}\rho^2|\nabla\lambda|^2 \end{aligned}$$

Therefore, $\quad \Delta F \leq -C_{12}\psi' u^\lambda e^{-\alpha\rho} + \psi u^\lambda \left(C_{21}\varepsilon_0^{-1}\rho^4|\nabla\lambda|^2 + (\log u)\Delta\lambda\right)$

Next, consider

$$\begin{aligned} &(\log u)\Delta\lambda \\ =\;& \log u\left(\Delta\lambda_1 + (\Delta\lambda_2)e^{-\delta_0\rho} - (1-\lambda_2)\Delta e^{-\delta_0\rho} + 2 < \nabla\lambda_2, \nabla e^{-\delta_0\rho} >\right) \\ =\;& \log u\left(\Delta\lambda_1 + e^{-\delta_0\rho}\Delta\lambda_2 - (1-\lambda_2)\delta_0 e^{-\delta_0\rho}(\delta_0 - \Delta\rho) + 2 < \nabla\lambda_2, -\delta_0 e^{-\delta_0\rho}\nabla\rho >\right) \\ \leq\;& C_{22}\rho\left(|\Delta\lambda_1| + e^{-\delta_0\rho}|\Delta\lambda_2| + e^{-\delta_0\rho}|\nabla\lambda_2|\right) - (1-\lambda_2)\delta_0 e^{-\delta_0\rho}(\delta_0 - \Delta\rho)\log u \\ \leq\;& C_{22}\left(C_{14}e^{-C_{15}\rho}(1-\lambda_1)^{1-\frac{\varepsilon_1}{2}} + e^{-\delta_0\rho}C_{14}e^{-C_{15}\rho}(1-\lambda_2)^{1-\varepsilon_1}\right) - (1-\lambda_2)C_{25}\rho e^{-\delta_0\rho} \end{aligned}$$

choosing $\delta_0 \leq \frac{1}{2}a(n-1) \leq \frac{1}{2}\Delta\rho$. Since $1-\lambda_1 = (1-\lambda_2)^2$, we have

$$\begin{aligned} &(\log u)\Delta\lambda \\ \leq\;& C_{22}\left(C_{14}e^{-C_{15}\rho}(1-\lambda_2)^{2-\varepsilon_1} + C_{14}e^{-C_{15}\rho}e^{-\delta_0\rho}(1-\lambda_2)^{1-\varepsilon_1}\right) - C_{25}\rho e^{-\delta_0\rho}(1-\lambda_2) \\ \leq\;& C_{23}e^{-C_{24}\rho}(1-\lambda_2)^{1-\varepsilon_1} - C_{25}\rho(1-\lambda_2)e^{-\delta_0\rho} \end{aligned}$$

Next, consider for $\rho \geq R_0$ large enough depending on $\varepsilon_0$,

$$\begin{aligned} &C_{21}\varepsilon_0^{-1}\rho^4|\nabla\lambda|^2 \\ \leq\;& C_{21}\varepsilon_0^{-1}\rho^4\left(|\nabla\lambda_1|^2 + |\nabla\lambda_2|^2 e^{-2\delta_0\rho} + (1-\lambda_2)^2|\nabla e^{-\delta_0\rho}|^2\right) \\ =\;& C_{21}\varepsilon_0^{-1}\rho^4|\nabla\lambda_1|^2 + C_{21}\varepsilon_0^{-1}\rho^4 e^{-2\delta_0\rho}|\nabla\lambda_2|^2 + C_{21}\varepsilon_0^{-1}\rho^4(1-\lambda_2)^2\delta_0^2 e^{-2\delta_0\rho} \\ \leq\;& C_{21}\varepsilon_0^{-1}\rho^4\left(C_{14}e^{-C_{15}\rho}(1-\lambda_2)^{2-\varepsilon_1} + C_{14}e^{-C_{15}\rho}(1-\lambda_2)^{1-\varepsilon_1}e^{-2\delta_0\rho}\right) \\ &+ C_{26}\rho^4\varepsilon_0^{-1}e^{-2\delta_0\rho}(1-\lambda_2)^2 \\ \leq\;& C_{26}e^{-C_{27}\rho}(1-\lambda_2)^{1-\varepsilon_1} + C_{26}\rho^4\varepsilon_0^{-1}e^{-2\delta_0\rho}(1-\lambda_2)^2 \end{aligned}$$

We have

$$\Delta F \le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + \psi u^\lambda \left(C_{28}e^{-C_{29}\rho}(1-\lambda_2)^{1-\varepsilon_1} - C_{30}(1-\lambda_2)e^{-\delta_0\rho}\right).$$

Choose $\delta_0 < \frac{1}{2}C_{29}$, let $C_{31} = C_{29} - \delta_0 > 0$. Then, $\Delta F \le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + \psi u^\lambda e^{-\delta_0\rho}\left(C_{28}e^{-C_{31}\rho}(1-\lambda_2)^{1-2\varepsilon_1} - C_{30}(1-\lambda_2)\right)$. Then, for $x \in T(\frac{3\pi}{16}, R_0)$ such that $C_{28}e^{-C_{31}\rho}(1-\lambda_2)^{1-2\varepsilon_1} \le C_{30}(1-\lambda_2)$, we have $\Delta F \le 0$. So, we consider for $x \in T(\frac{3\pi}{16}, R_0)$ such that $C_{28}e^{-C_{31}\rho}(1-\lambda_2)^{1-2\varepsilon_1} \ge C_{30}(1-\lambda_2)$. Then we have

$$1-\lambda_2 \le C_{28}C_{30}^{-1}e^{-C_{31}\rho}(1-\lambda_2)^{1-2\varepsilon_1} \text{ and hence } 1-\lambda_2 \le C_{33}e^{-\frac{1}{2}C_{31}\varepsilon_1^{-1}\rho}$$

Then,

$$\begin{aligned}
\Delta F &\le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + C_{28}\psi u^\lambda e^{-\delta_0\rho}e^{-C_{31}\rho}C_{32}^{1-2\varepsilon_1}e^{-\frac{1}{2}C_{31}\varepsilon_1^{-1}(1-2\varepsilon_1)\rho} \\
&\le -C_{12}\psi' u^\lambda e^{-\alpha\rho} + C_{33}u^\lambda e^{-\delta_0\rho - C_{31}\rho}e^{-\frac{1}{2}C_{31}\varepsilon^{-1}\rho}e^{C_{31}\rho} \\
&\le u^\lambda\left(C_{33}e^{-\frac{1}{2}\varepsilon_1^{-1}C_{31}\rho} - C_{34}\rho^{-2}e^{-\alpha\rho}\right)
\end{aligned}$$

Choose $\varepsilon_1 > 0$ small enough such that $\alpha < \frac{1}{2}C_{31}\varepsilon_1^{-1}$, i.e. $C_{31}\varepsilon_1^{-1} - \alpha > 0$. Then,

$$\Delta F \le u^\lambda e^{-C_{31}\varepsilon_1^{-1}\rho}\left(C_{33} - C_{34}\rho^{-2}e^{(C_{31}\varepsilon_1^{-1}-\alpha)\rho}\right) \le 0$$

for $\rho \ge R_0$ large enough.

Therefore, we have on $T(\frac{3\pi}{16}, R_0)$, $\Delta F = \Delta\left(u^\lambda \cdot \psi(\rho_1 - e^{-\alpha\rho})\right) \le 0$.

On $C(\frac{3\pi}{16}) \cap \partial B_0(R_0)$, $F = u^\lambda\psi(\rho_1 - e^{-\alpha\rho}) \ge u^\lambda \ge \widetilde{C}u \ge C$ using Harnack Inequality.

On $\partial C(\frac{3\pi}{16})\backslash B_0(R_0)$, $v \le u^\lambda \le u^\lambda C_{36}\psi(\rho_1 - e^{-\alpha\rho}) = C_{36}F$.

Therefore, $C_{36}F \ge v$ on $\partial T(\frac{3\pi}{16}, R_0)$, hence on $T(\frac{3\pi}{16}, R_0)$.

Finally, on $T(\frac{\pi}{8}, R_0)$, $\lambda \equiv 1$,

$$v \le C_{36}F = C_{36}u^\lambda\psi(\rho_1 - e^{-\alpha\rho}) \le C_{37}u^\lambda = C_{37}u$$

By Harnack Inequality, $v \le C_{37}u$ on $T(\frac{pi}{8}, 1)$. Hence $\frac{u}{v} \ge C_{38}$. Similarly, we have $\frac{v}{u} \ge C_{38}$. That is,

$$\frac{1}{C_1} \le \frac{u}{v} \le C_1$$

$$\frac{1}{C_1}\frac{u(0')}{v(0')} \le \frac{u}{v} \le C_1\frac{u(0')}{v(0')}$$

on $T(\frac{\pi}{8}, 1)$. □

COROLLARY **3.2 (Harnack Inequality at Infinity).** *Let $u$, $v$ be positive harmonic functions on $C = C_0(\frac{\pi}{4})$, continuous on $\overline{C}$ with $u = v = 0$ on $\overline{C} \cap M(\infty)$. Then, there exists constant $K$ depending only on $n$, $a$, $b$ such that*

$$\sup_{T_0(\frac{\pi}{8},1)} \frac{u}{v}(x) \le K \inf_{T_0(\frac{\pi}{8},1)} \frac{u}{v}(x)$$

## 3.3 THE KERNEL FUNCTION

DEFINITION **3.1.** *Let 0 be a fixed point in $M$, a kernel function $K$ at $\xi \in M(\infty)$ is a positive harmonic function on $M$ such that $K(0) = 1$, $K$ extends continuously to zero on $M(\infty)\backslash\{\xi\}$.*

*Existence of kernel functions:* Let $\{u_k\}$ be a sequence of positive harmonic functions defined on $C(\frac{\pi}{4})$. Suppose $u_k$ vanishes on $\overline{C(\frac{\pi}{4})} \cap M(\infty)$, $u_k(0') \le C$, $0' = exp_0 v_0$, $C$ constant. By corollary 3.1(see the next section), $\{u_k\}$ is uniformly bounded and of uniform exponential decay in $T(\frac{\pi}{8}, 1)$. Therefore, $\{u_k\}$ has a convergence subsequence with limit $u \in C^\infty\left(T(\frac{\pi}{8}, 1)\right) \cap C^0\left(\overline{T(\frac{\pi}{8}, 1)} \cap M(\infty)\right)$, $u \equiv 0$ on $\overline{T(\frac{\pi}{8}, 1)} \cap M(\infty)$. Now, consider $\{y_i\}$ is a sequence in $M$ converging to $\xi \in M(\infty)$. Then, for $h_i(x) = \dfrac{G(y_i, x)}{G(y_i, 0)}$, $\{h_i\}$ is a sequence of harmonic functions which converge to $h_\xi$, which is also positive harmonic in $M$. Using the above, we have $h_\xi = 0$ on $M(\infty)\backslash\{\xi\}$. Therefore, $h_\xi$ extends continuously to zero on $M(\infty)\backslash\{\xi\}$.

PROPOSITION **3.1.** *Let $\xi \in M(\infty)$, $\exists!$ kernel functions $K_\xi$ at $\xi$.*

*Proof.* Suppose $f$, $g$ are kernel functions at $\xi$ such that $f(0) = g(0) = 1$. Let $\sigma(t)$ be the geodesic ray from 0 to $\xi$, $C(t)$ be the cone with vertex $\sigma(t)$, angle $\frac{\pi}{4}$, direction $\sigma'(t)$. Let $\widetilde{C}(t)$ be the cone $M\backslash C(t)$. By Comparison Principle, $\exists C > 0$ such that on $\widetilde{C}(t)\backslash B(\sigma(t), 1)$,

$$\frac{1}{C} = \frac{1}{C}\frac{g(0)}{f(0)} \leq \frac{g(x)}{f(x)} \leq C\frac{g(0)}{f(0)} = C$$

Hence, $g(x) \leq Cf(x)$ on $M$ by letting $t \to \infty$.

Let $a = \inf\{C > 0 : Cf(x) \geq g(x), x \in M\}$. We have, either $af \equiv g$ or $af > g$ by Maximum Principle.

If $af \equiv g$, $af(0) = g(0)$ implies $a = 1$ and $f \equiv g$ on $M$. Hence, we can suppose $af > g$. Therefore, $a = af(0) > g(0) = 1$, which implies $a - 1 > 0$. Let $h = \frac{af-g}{a-1}$, which is a kernel function at $\xi$. Therefore $C^2h \geq g$, $C^2(af - g) \geq (a-1)g$, hence $\frac{aC^2}{a+C^2-1}f \geq g$. By definition, $\frac{aC^2}{a+C^2-1} \geq a$, therefore $C^2 \geq a + C^2 = 1$ and $a - 1 \leq 0$. That is, $f \equiv g$. □

## 3.4 The Main Theorem

We are now ready to give partial results of the main theorem:

PROPOSITION **3.2.** *There exists a continuous surjective map $\Phi : \widetilde{M} \to M(\infty)$.*

*Proof.* Let $\mathcal{Y} = [\{y_i\}] \in \widetilde{M}$, then, for $\{y_i\} \in \mathcal{Y}$, $\{h_{y_i}\} \to h$. We want to show that $\{y_i\} \to \xi \in M(\infty)$.

Consider if $\{\widetilde{y_i}\}$ is a conv. subseq. of $\{y_i\}$, we have, $\{\widetilde{h_i}\} \to h$ which is a kernel function at $\widetilde{\xi} = \lim \widetilde{y_i}$. Then, if $\{\bar{y_i}\}$ is another conv. subseq. of $\{y_i\}$, $\{\bar{h_i}\} \to h$ which is a kernel function at $\bar{\xi} = \lim \bar{y_i}$

Thus, $\widetilde{\xi} = \bar{\xi}$, hence, $\{y_i\}$ have an unique limit $\xi \in M(\infty)$.

So, we define $\Phi(\mathcal{Y}) = \xi$. To prove that $\Phi$ is well defined, consider $\{y_i^1\}, \{y_i^2\} \in \mathcal{Y}$, such that $\{y_i^1\} \to \xi_1, \{y_i^2\} \to \xi_2, \{h_i^1\} \to h, \{h_i^2\} \to h$

Consider the sequence $\{y_1^1, y_1^2, y_2^1, y_2^2, \cdots\} \in \mathcal{Y}$. By above, $\xi_1 = \xi_2$, hence, $\Phi$ is well defined.

To prove $\Phi$ is continuous, let $\mathcal{Y}_i \to \mathcal{Y}$ in $\widetilde{M}$, i.e., $\rho(\mathcal{Y}_i, \mathcal{Y}) \to 0$, where,

$$\rho(\mathcal{Y}_i, \mathcal{Y}) = \sup_{B(0,1)} |h_{\mathcal{Y}_i} - h_{\mathcal{Y}}|,$$

and, $h_{y_k^i} \to h_{\mathcal{Y}_i}, h_{y_k} \to h_{\mathcal{Y}}$ if $\{y_k^i\} \in \mathcal{Y}_i, \{y_k\} \in \mathcal{Y}$

Suppose $\{y_k^i\} \to \xi_i, \{y_k\} \to \xi$, then, $\Phi(\mathcal{Y}_i) = \xi_i, \Phi(\mathcal{Y}) = \xi$

Let $\{\xi_{i_n}\}$ be a conv. subseq. of $\{\xi_i\}$, since $h_{i_n}(0) = 1$ for all $i_n$,

by Interior Harnack Inequality, there exists convergence subsequence of $h_{i_n}$, therefore, passing to subsequence, $h_{i_n} \to g$ uniformly on compact sets. Then, $g$ is positive harmonic and $g(0) = 1$.

Suppose $\lim \xi_{i_n} = \eta$, let $C(\frac{\pi}{4})$ be a cone in $M$ such that $\eta \notin G = C(\frac{\pi}{4}) \cap M(\infty)$, by Cor. 3.1, $\forall \epsilon > 0$, $\exists C_1$, $C_2$ independent of $\eta$ such that $h_{i_n}(x) \leq c_1 e^{-c_2\rho(x)} h_{i_n}(0)$ on $T(\frac{\pi}{4} - \epsilon, 1)$. Then, let $i_n \to \infty$, we get $g = 0$ on $G$, hence on $M(\infty)\backslash\eta$. Therefore, $h_{i_n} \to g$ is a kernel function at $\eta$.

Since $h_i \to h$ is a kernel function at $\xi$, hence $h = g$, $\xi = \eta$, therefore, $\{\xi_{i_n}\} \to \xi$, i.e., $\Phi(\mathcal{Y}_i) \to \Phi(\mathcal{Y})$, hence, $\Phi$ is continuous.

To prove that $\Phi$ is surjective, let $\xi \in M(\infty)$, take $\{y_i\}$ a sequence in $M$ such that $\{y_i\} \to \xi$. Then, there exists subsequence of $\{y_i\}$, say, $\{y_{i_n}\}$ such that $\{h_{i_n}\} \to h$ a kernel function at $\xi$. Therefore, $\{y_{i_n}\}$ is a fundamental sequence such that $\Phi([\{y_{i_n}\}]) = \xi$. $\square$

LEMMA **3.2.** *There exists positive constants $T_1$, $T_2$ depending only on $a$, $b$ such that if $t_1 \leq t + T_1$, $t_2 \geq t + T_2$,*

$$C_t\left(\frac{\pi}{8}\right) \cap M(\infty) \subset C_{t_1}\left(\frac{\pi}{4}\right) \cap M(\infty)$$

$$C_{t_2}\left(\frac{\pi}{4}\right) \subset C_t\left(\frac{\pi}{8}\right)$$

*Proof.* Let $\sigma(t)$ be a geodesic on $M$, $x_0 = \sigma(t)$, $x_1 = \sigma(t')$, $t < t'$, $T = \rho(x_0, x_1)$. Take $P \in \partial C_{t'}\left(\frac{\pi}{4}\right)$, $d_1 = \rho(P, x_1)$, $d_0 = \rho(P, x_0)$, $\alpha =$ angle between $x_0P$ and

$x_0x_1$. Using Toponogov Comparison Theorem,

$$\begin{aligned}\cos\alpha &\geq \coth ad_0 \coth aT - \frac{\cosh ad_1}{\sinh ad_0 \sinh aT} \\ &= \coth ad_0 \left[\coth aT - \frac{1}{\sinh aT}\cdot\frac{\cosh ad_1}{\cosh ad_0}\right]\end{aligned}$$

$$\cos\left(\pi - \frac{\pi}{4}\right) \geq \coth aT \coth ad_1 - \frac{\cosh ad_0}{\sinh aT \sinh ad_1}$$

$$-\frac{\sqrt{2}}{2}\sinh aT \sinh ad_1 \geq \frac{-1}{\cosh aT + \frac{\sqrt{2}}{2}\sinh aT \tanh ad_1}$$

We have $\cos\alpha \geq \coth ad_0 \left[\coth aT - \frac{1}{\sinh aT\left(\cosh aT + \frac{\sqrt{2}}{2}\tanh ad_1 \sinh aT\right)}\right].$

To prove the second inequality, it is sufficient to prove that for $t_2 \geq t + T_2$, $t_2' = t_2 - t$, $\forall P \in \partial C_{t_2}\left(\frac{\pi}{4}\right)$, $\alpha_p \leq \frac{\pi}{8} - \varepsilon$, i.e., $\cos\alpha_p \geq \cos(\frac{\pi}{8} - \varepsilon)$.

$$\begin{aligned}\cos\alpha_p &\geq \coth ad_0 \left[\coth at_2' - \frac{1}{\sinh at_2'\left(\cosh at_2' + \frac{\sqrt{2}}{2}\tanh ad_1 \sinh at_2'\right)}\right] \\ &\geq \coth at_2' - \frac{1}{\sinh at_2'\left(\cosh at_2' + \frac{1}{\sqrt{2}}\tanh ad_1 \sinh at_2'\right)} \\ &\geq \coth at_2' - \frac{1}{\sinh at_2' \cosh at_2'} \\ &= \frac{\cosh^2 at_2' - 1}{\sinh at_2' \cosh at_2'} = \frac{\sinh^2 at_2'}{\sinh at_2' \cosh at_2'} = \tanh at_2' \\ &\geq \tanh aT_2\end{aligned}$$

Therefore, choose $T_2$ such that $\tanh aT_2 \geq \cos(\frac{\pi}{8} - \varepsilon)$. Then, $\alpha_p \leq \frac{\pi}{8} - \varepsilon \leq \frac{\pi}{8}$.

To prove the first inequality, first note that for $T \leq T_1$, $d_0$, $d_1 > 0$ large enough, we have

$$\sin\alpha \geq \sin\alpha_b = \left(\sin\frac{3\pi}{4}\right)\left(\frac{\sinh bd_1}{\sinh bd_0}\right) \geq \left(\sin\frac{3\pi}{4}\right)\left(\frac{\sinh b(d_0 - T)}{\sinh bd_0}\right)$$

$$\frac{\sinh b(d_0 - T)}{\sinh bd_0} = \frac{e^{b(d_0-T)} - e^{-b(d_0-T)}}{e^{bd_0} - e^{-bd_0}} = \frac{e^{-bT_1} - e^{-b(2d_0-T_1)}}{1 - e^{-2bd_0}} \geq e^{-bT_1} - e^{-b(2d_0-T_1)}.$$

Then, for $T_1 > 0$, small enough fixed, $\exists d_1 > 0$ large enough such that $\sin\frac{3\pi}{4}\left(e^{-bT_1} - e^{-b(2d_0-T_1)}\right) \geq sin\frac{\pi}{8}$, i.e., $\sin\frac{3\pi}{4}/\sin\pi 8 \geq \frac{e^{bT_1}}{1-e^{-2b(d_0-T_1)}}$.

$$\text{Therefore, } \sin\alpha \geq \sin\frac{\pi}{8} \text{ and } \alpha \geq \frac{\pi}{8}$$

□

THEOREM **3.3.** *Let $u$, $v$ be positive harmonic functions in a cone $C$, continuous on $\overline{C}$ such that $u$ and $v$ equal 0 when restricted on $\overline{C}\cap M(\infty)$. Then, the quotient $\frac{u}{v}$ has a $C^\alpha$-extension to the interior of $\overline{C}\cap M(\infty)$, where $\alpha$ depends only on $n$, $a$ and $b$.*

*Proof.* Fix $\xi_0 \in \left(\overline{C}\cap M(\infty)\right)^\circ$, let $x_0$ be the vertex of $C$. Let $\sigma(t)$ be the geodesic ray $\overline{x_0\xi_0}$, $C_i = C_{\sigma(iT_2)}(\frac{\pi}{4})$, $T_2$ determined by the previous lemma. Then, for $i$ large enough, $\{C_i \cap M(\infty)\}$ is a nested sequence of open neighborhoods of $\xi_0$ in $\overline{C}$ with $\cap_{i\in\mathbb{N}}(C_i \cap M(\infty)) = \xi_0$.

Let $\varphi = \frac{u}{v}$, define

$$\underline{\varphi_i} = \inf_{y\in C_i}\varphi(y), \overline{\varphi_i} = \sup_{y\in C_i}\varphi(y)$$

Using the Harnack Inequality at Infinity, we have

$$\sup_{y\in C_{i+1}}\frac{u_i}{v}(y) \leq K\inf_{y\in C_{i+1}}\frac{u_i}{v}(y)$$

$$\sup_{y\in C_{i+1}}\left(\frac{u}{v} - \underline{\varphi_i}\right) \leq K\inf_{y\in C_{i+1}}\left(\frac{u}{v} - \underline{\varphi_i}\right)$$

i.e., $\overline{\varphi_{i+1}} - \underline{\varphi_i} \leq K(\underline{\varphi_{i+1}} - \underline{\varphi_i})$. Similarly, defined $v_i = \overline{\varphi_i}v - u$, we have $\overline{\varphi_i} - \underline{\varphi_{i+1}} \leq K(\overline{\varphi_i} - \overline{\varphi_{i+1}})$.

Define $\omega_i = \overline{\varphi_i} - \underline{\varphi_i}$, then, $\omega_{i+1} + \omega_i \leq K(\omega_i - \omega_{i+1})$. Hence, $\omega_{i+1} \leq \frac{K-1}{K+1}\omega_i = \mu\omega_i$.

Then, $\omega_i \leq \mu^i\omega_0 = \mu^i(\overline{\varphi_0} - \underline{\varphi_0}) \leq \mu^i \sup\varphi$.

Therefore, we can extend $\varphi$ to $\overline{C}$ by defining for $\xi_0 \in \overline{C}$, $\varphi(\xi_0) = \lim\varphi(y_i)$.

Now let $\theta_i$ be the maximum angle subtends by $C_i$ at $x_0$. We first claim that

$\theta_i \geq e^{(-Cb)i}$ for $i$ large enough, $C$ constant. If true, for $y$, $y'$ sufficiently distant from $x_0$, we have

$$|\varphi(y) - \varphi(y')| \leq \mu^i \varphi(x_0) = e^{i \log u} \varphi(x_0)$$

Let $\alpha = \frac{1}{Cb} \log \frac{1}{mu}$, hence, $\log \mu = -Cb\alpha$, we have

$$|\varphi(y) - \varphi(y')| \leq e^{-Cbi\alpha} \varphi(x_0) \leq \theta_0^\alpha \varphi(x_0) \leq [\angle_{x_0}(y, y')]^\alpha \varphi(x_0)$$

Hence, $\varphi = \frac{u}{v}$ is $C^\alpha$ in $\left(\overline{C} \cap M(\infty)\right)^\circ$.

We now prove that $\theta_i \geq e^{-Cbi}$. We have $\theta_i = \sup\limits_{p \in \partial C_i(\frac{\pi}{4})} \theta_p$, $\theta_p$ = angle between $\overline{x_0 p}$ and $\overline{x_0 x_i}$, $x_i = \sigma(iT2)$, $d_0 = \rho(x_0, p)$, $d_1 = \rho(x_i, p)$, angle between $\overline{x_i p}$ and $\overline{x_0 x_i} = \frac{3\pi}{4}$. Then,

$$\cos \theta_p \leq \frac{\cosh bT \cosh bd_0 - \cosh bd_1}{\sinh bT \sinh bd_0}$$

$\theta_p + \alpha_p + \frac{3\pi}{4} < \pi$, hence $\alpha_p < \frac{\pi}{4}$. Therefore,

$$\cos \frac{\pi}{4} < \cos \alpha_p \leq \frac{\cosh bd_0 \cosh bd_1 - \cosh bT}{\sinh bd_0 \sinh bd_1}$$

$$\cosh bd_1 \geq \frac{\sinh bd_0 \sinh bd_1}{\sqrt{2} \cosh bd_0} + \frac{\cosh bT}{\cosh bd_0}$$

Then,

$$\begin{aligned}
\cos \theta_p \sinh bT \sinh bd_0 &\leq \cosh bT \cosh bd_0 - \cosh bd_1 \\
&\leq \cosh bT \cosh bd_0 - \frac{\sinh bd_0 \sinh bd_1}{\sqrt{2} \cosh bd_0} - \frac{\cosh bT}{\cosh bd_0} \\
&= \cosh bT \left( \frac{\cosh^2 bd_0 - 1}{\cosh bd_0} \right) - \frac{\sinh bd_0 \sinh bd_1}{\sqrt{2} \cosh bd_0} \\
&= \cosh bT \frac{\sinh^2 bd_0}{\cosh bd_0} - \frac{\sinh bd_0 \sinh bd_1}{\sqrt{2} \cosh bd_0}
\end{aligned}$$

Therefore,

$$\cos \theta_p \leq \coth bT \tanh bd_0 - \frac{1}{\sqrt{2}} \frac{\sinh bd_1}{\cosh bd_0} \frac{1}{\sinh bT}$$

For $T$ fixed, $d_0 + T > d_1 > d_0 - T \implies \frac{\sinh bd_1}{\sinh bd_0} \to 1$ as $d_0 \to \infty$. Let $\theta_\infty = \lim_{p\to\infty} \theta_p$. Then,

$$1 - \frac{\theta_\infty^2}{2} \leq \cos\theta_\infty \leq \coth bT - \frac{1}{\sqrt{2}\sinh bT} = \frac{1+e^{-2bT}}{1-e^{-2bT}} - \frac{\sqrt{2}e^{-bT}}{1-e^{-2bT}}$$

Hence,

$$\frac{\theta_\infty^2}{2} \geq \frac{\sqrt{2}e^{-bT} - 2e^{-2bT}}{1-e^{-2bT}} \geq \sqrt{2}e^{-bT} - 2e^{-2bT} \geq e^{-CbT}$$

□

We can now turn to our main theorem of this chapter, which shows that $M(\infty) \cong \widetilde{M}$. Also, the $C^\alpha$ structure of $M(\infty)$ carries to $\widetilde{M}$ via this 1-1 correspondence.

THEOREM **3.4.** *The natural map $\Phi : \widetilde{M} \to M(\infty)$ is a homeomorphism. In particular, $\widetilde{M}$ carries a natural $C^\alpha$ manifold structure for some $\alpha > 0$, $\alpha$ depending on $n$, $a$ and $b$. The map $\Phi^{-1} : M(\infty) \to \widetilde{M}$ is $C^\alpha$ relative to the Martin metric $\rho$. Where for $\mathcal{Y}$, $\mathcal{Y}' \in \widetilde{M}$, $\rho(\mathcal{Y}, \mathcal{Y}') = \sup_{B_0(1)} |h_{\mathcal{Y}}(x) - h_{\mathcal{Y}'}(x)|$.*

*Proof.* Let $\mathcal{Y} = [\{y)_i\}]$, $\mathcal{Z} = [\{z_i\}]$. Suppose $\Phi(\mathcal{Y}) = \Phi(\mathcal{Z})$. Then, if

$$h_{\mathcal{Y}}(x) = \lim_{i\to\infty} \frac{G(y_i,x)}{G(y_i,0)}, h_{\mathcal{Z}}(x) = \lim_{i\to\infty} \frac{G(z_i,x)}{G(z_i,0)}, \{y_i\} \to \xi, \{z_i\} \to \xi$$

$h_{\mathcal{Y}}$, $h_{\mathcal{Z}}$ are kernel functions at $\xi$, hence $h_{\mathcal{Y}} = h_{\mathcal{Z}}$, $\mathcal{Y} = \mathcal{Z}$, i.e., $\Phi$ is $1-1$.

To prove that $\Phi^{-1}$ is $C^\alpha$, let $\xi$, $\xi' \in M(\infty)$, $C_1$, $C_2$ be cones such that $\xi$, $\xi' \in C_1^\circ$, $C_1 \subset C_2^\circ$, $C_1 \neq C_2^\circ$, $0'$ be the vertex of $C_1$, $C_2$, $\rho(0, 0') = 2$. Then, for $x \in B_0(1)$, $G(y, x)$ is well-defined on $C_2$. Then

$$\begin{aligned} & \left| \lim_{i\to\infty} \frac{G(y_i,x)}{G(y_i,0)} - \lim_{i\to\infty} \frac{G(z_i,x)}{G(z_i,0)} \right| \\ \leq\ & k_1 \sup_{y\in C_2} \frac{G(y,x)}{G(y,0)} \angle_{0'}^\alpha(\xi,\xi') \\ \leq\ & k_2 \frac{G(0,x)}{G(0,0')} \angle_{0'}^\alpha(\xi,\xi') \leq k_3 \angle_{0'}^\alpha(\xi,\xi') \end{aligned}$$

$$\rho(\Phi^{-1}(\xi), \Phi^{-1}(\xi')) = \rho(\mathcal{Y}, \mathcal{Z}) = \sup_{B_0(1)} |K(x,\xi), K(x,\xi')| \leq k_3 \angle_0^\alpha(\xi,\xi')$$

Therefore, $\Phi^{-1}$ is $C^\alpha$ w.r.t. $\rho$. □

# Chapter 4

# Positive Harmonic Functions on Product of Manifolds

## 4.1 Splitting Theorem

In this chapter, we will discuss a result of Freire [Fre], which shows that on a product of Riemannian manifolds $M = M_1 \times M_2$, minimal positive harmonic functions can be splitted into product of minimal positive eigenfunctions on each factor, we require only each factor $M_1$ and $M_2$ are complete, non-compact Riemannian manifolds with Ricci curvatures bounded from below.

On a complete, non-compact Riemannian manifold $M$, denote

$$\lambda_0(M) = -\inf\{\frac{\int_M |\nabla\phi|^2}{\int_M |\phi|^2} : \phi \in C^\infty(M) \text{ has compact support}\}.$$

Then, for each $\lambda \geq \lambda_0$, there exists positive solutions to the eigenvalue problem

$$\Delta\varphi = \lambda\varphi.$$

For $\lambda > \lambda_0$, the $\lambda$-Green's function $G^\lambda(x, y) > 0$ exists for the $\lambda$-eigenvalue problem (See [Sul2]). Then, for $\lambda \geq \lambda_0(M)$, we define

$\widetilde{M_1^\lambda} = \{f > 0 : \Delta f = \lambda f$ and if $0 < g \leq f, \Delta g = \lambda g,$ we have $g = kf, k$ constant.$\}$

The following theorem was proved by Freire.

THEOREM **4.1.** *Let $M = M_1 \times M_2$ be a Riemannian product, where $M_1$ and $M_2$ are complete, non-compact, with Ricci curvature bounded below. Then, we have,*
*(i) Each minimal positive harmonic function $f$ on $M$ splits as a product*

$$f(x) = K^{\lambda_1}(x^1)K^{\lambda_2}(x^2),$$

*where $\lambda_i \geq \lambda_0(M_i)$, $K^{\lambda_i} \in \widetilde{M_1^{\lambda_i}}(M_i)$ for $i = 1, 2$, and $\lambda_1 + \lambda_2 = 0$.*
*(ii) Conversely, each product as above is a minimal positive harmonic function on $M$.*

We will prove this via constructing a parabolic Martin boundary for the Riemannian halfspace.

## 4.2 Riemannian Halfspace and the parabolic Martin boundary

Let $M$ be a complete, non-compact Riemannian manifold without boundary. We define the Riemannian halfspace to be

$$\mathcal{H} = M \times (-\infty, 0],$$

and denote

$$\mathcal{H}' = M \times (-\infty, 0).$$

Define the harmonic sheaf $P$ to be

$$P(U) = \{u(\overline{x}) \in C(U) : u|_{\mathcal{H}'} \in C^2(U \cap \mathcal{H}') \text{ and } \Delta u = \frac{\partial u}{\partial t} \text{ in } U \cap \mathcal{H}'\}$$

for any $U \subset \mathcal{H}$ open. For $u(\overline{x}) \in P(U)$, $u$ is said to be a parabolic function on $U$. Together with $\mathcal{H}$, the pair $(\mathcal{H}, P)$ is called the (Bauer) harmonic space. Throughout this chapter, we will denote by $G(\overline{x}, \overline{y})$ the Green's function on $(\mathcal{H}, P)$:

$$G(\overline{x}, \overline{y}) = \begin{cases} h(t-s, x, y), & t > s \\ 0, & t \leq s, \end{cases}$$

where $k_h(t,x,y)$ is the heat kernel of $M$, $\overline{x}=(x,t)\in\mathcal{H}$ and $\overline{y}=(y,s)\in\mathcal{H}'$. Denote $H(x,t;y,s)=\dfrac{G(\overline{x},\overline{y})}{G(\overline{x_0},\overline{y})}$. Then, for a sequence $\{\overline{y_i}\}$ in $\mathcal{H}'$, we said that it is fundamental iff it has no accumulation point in $\mathcal{H}$ and $H(\overline{x},\overline{y_i})\to H(\overline{x},\xi)$ a non-negative $C^2$ solution of the heat equation, uniformly on compact sets of $\mathcal{H}'$. $H(\overline{x},\xi)$ is called the Martin kernel. Two fundamental sequences are said to be equivalent iff the limiting Martin kernels are the same. In the case that $s_i\to 0$ and $\{y_i\}$ bounded in $M$, where $\overline{y_i}=(y_i,s_i)$, we also define $H(\overline{x},\xi)$ to be 0. We define the parabolic Martin boundary, denoted by $\widetilde{\mathcal{H}}$, to be the set of all equivalent classes of fundamental sequences, and the parabolic Martin compactification $\overline{\mathcal{H}}=\mathcal{H}'\bigcup\widetilde{\mathcal{H}}$. Choose a smooth function $0<\phi<1$ on $\mathcal{H}$ such that $\int_{\mathcal{H}}\phi(x,t)dxdt<\infty$, define

$$d(\overline{y_1},\overline{y_2})=\int_{\mathcal{H}}\min\{1,|H(\overline{x},\overline{y_1})-H(\overline{x},\overline{y_2})|\}\phi(\overline{x})d\overline{x}$$

for $\overline{y_1}$, $\overline{y_2}\in\overline{\mathcal{H}}$. Then, $(\overline{\mathcal{H}},d)$ forms a compact metric space and the topology induced by $d$ coincides with the original topology of $\mathcal{H}'$.

DEFINITION **4.1.** *The minimal parabolic Martin boundary is defined to be*

$$\widetilde{\mathcal{H}_1}=\{0\le u\in P(\mathcal{H}):u(\overline{x_0})=1\ \textit{and}\ 0\le v\le u, v\in P(\mathcal{H})\implies v=ku, k\ \textit{constant.}\}$$

THEOREM **4.2 (Parabolic Martin representation).** *Let $u(\overline{x})\ge 0$, $u\in P(\mathcal{H})$. Then there exists a unique finite Borel measure $\mu_u$ on $\widetilde{\mathcal{H}_1}$ such that for $\overline{x}\in\mathcal{H}$ and $\mu_u(\widetilde{\mathcal{H}_1})=u(x_0,0)$,*

$$u(\overline{x})=\int_{\widetilde{\mathcal{H}_1}}H(\overline{x},\xi)d\mu_u(\xi).$$

For a proof of this theorem, see [Fre].

## 4.3 Splitting of parabolic Martin kernels

In this section, we will prove the first part of our main theorem 4.1, which states that minimal positive harmonic functions can be splitted into product of minimal positive eigenfunctions of each factor $M_1$ and $M_2$.

THEOREM **4.3.** *Assume Ricci curvature of $M$ is bounded below, then*

$$\widetilde{\mathcal{H}_1} = \{\xi \in \widetilde{\mathcal{H}} : H(\overline{x}, \xi) = e^{\lambda t} K^{\lambda}(x, \xi'), \lambda \geq \lambda_0(M), \xi' \in \widetilde{M_1^{\lambda}}\}$$

*Proof.* Let $u(x,t)$ be minimal positive parabolic normalized at $(x_0, 0)$. For $s > 0$, let $u_s(x,t) = u(x, t-s)$, then, $u_s(x,t) \in P(\mathcal{H})$. By parabolic Harnack Inequality[L-Y], $u_s(x,t) \leq k_s u(x,t)$, $(x,t) \in \mathcal{H}$. Since $u$ minimal, we have $u_s(x,t) = ku(x,t)$ and $u(x_0, 0) = 1$ implies $k = u(x_0, -s)$. Therefore,

$$u_s(x,t) = u(x_0, -s) u(x,t).$$

Let $\phi(t) = u(x_0, -t) \in C([0,\infty))$. Then, $u_{s_1+s_2}(x,t) = u(x_0, -s_2) u(x, t - s_1) = u(x_0, -s_2) u(x_0, -s_1) u(x,t)$. Hence, $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2)$ $\forall t_1, t_2 > 0$. Also, $\phi(0) = u(x_0, 0) = 1$, therefore, $\phi(t) = e^{-\lambda t}$ for some $\lambda$.
Therefore, $u(x, t-s) = e^{-\lambda s} u(x,t)$, $u(x, -s) = e^{-\lambda s} u(x, 0)$. Hence,

$$u(x,t) = e^{\lambda t} u(x, 0).$$

Now, since $u$ parabolic, we have

$$\Delta \left(e^{\lambda t} u(x,0)\right) = \frac{\partial}{\partial t} \left(e^{\lambda t} u(x,0)\right).$$

Hence, $e^{\lambda t} \Delta u(x,0) = \lambda e^{\lambda t} u(x,0)$, i.e., $\Delta u(x,0) = \lambda u(x,0)$. Now we have $u(x,t)$ minimal, let $v(x) \leq u_0(x) = u(x,0)$ be positive parabolic. Then, $\Delta \left(e^{\lambda t} v(x)\right) = e^{\lambda t} \Delta v(x) = \lambda e^{\lambda t} v(x) = \frac{\partial}{\partial t} \left(e^{\lambda t} v(x)\right)$. We have $e^{\lambda t} v(x) \leq e^{\lambda t} u_0(x) = u(x,t)$, therefore $e^{\lambda t} v(x) = k e^{\lambda t} u_0(x)$, i.e., $v(x) = k u_0(x)$. Hence, the minimality of $u(x)$ implies the minimality of $u_0(x)$, i.e., $u_0(x) \in \widetilde{M_1^{\lambda}}$. Now let $u(x,t) = e^{\lambda t} \phi_{\lambda}(x)$, $\phi_{\lambda}(x) \in \widetilde{M_1^{\lambda}}$, then $u(x,t)$ is easily seen to be minimal parabolic, we will now prove that it can be written in the form $e^{\lambda t} K^{\lambda}(x, \xi')$. First, we will define the following maps:

DEFINITION **4.2.**
$p : \widetilde{\mathcal{H}_1} \to [\lambda_0(M), \infty), \xi \mapsto \lambda$ *such that* $H(x,t;\xi) = e^{\lambda t}\phi(x), \phi \in \widetilde{M_1^{\lambda}}$,
$b_{\lambda} : \{\xi \in \widetilde{\mathcal{H}_1} : p(\xi) = \lambda\} \to \widetilde{M_1^{\lambda}}, \xi \mapsto \xi'$ *iff* $H(x,t;\xi) = e^{\lambda t} K^{\lambda}(x, \xi'), (x,t) \in \mathcal{H}$.

Now, given $e^{\lambda t}\phi_\lambda(x) = \int_{\widetilde{\mathcal{H}_1}} H(x,t;\xi)d\mu(\xi)$, $\mu(\widetilde{\mathcal{H}_1}) = 1$, we claim that for $\lambda_1 < \lambda$, $\mu\{p(\xi) < \lambda_1\} = 0$. Consider

$$\begin{aligned}
e^{\lambda t}\phi_\lambda(x_0) &= \int_{\widetilde{\mathcal{H}_1}} H(x_0,t;\xi)d\mu(\xi) \\
&\geq \int_{\{p(\xi)<\lambda_1\}} e^{p(\xi)t}\phi_{p(\xi)}(x_0)d\mu(\xi) = \int_{\{p(\xi)<\lambda_1\}} e^{p(\xi)t}d\mu(\xi) \\
&\geq e^{\lambda_1 t}\mu\{p(\xi) < \lambda_1\}
\end{aligned}$$

Hence $\mu\{p(\xi) < \lambda_1\} \leq e^{(\lambda-\lambda_1)t} \to 0$ as $t \to -\infty$. Therefore,

$$e^{\lambda t}\phi_\lambda(x) = \int_{\{p(\xi)\geq\lambda\}} e^{p(\xi)t}d\mu(\xi).$$

Now,

$$\begin{aligned}
1 = \phi_\lambda(x_0) &= \int_{\{p(\xi)\geq\lambda\}} e^{p(\xi)t}e^{-\lambda t}d\mu(\xi) \\
&= \int_{\{p(\xi)=\lambda\}} e^{(p(\xi)-\lambda)t}d\mu(\xi) + \int_{\{p(\xi)>\lambda\}} e^{(p(\xi)-\lambda)t}d\mu(\xi) \\
\text{Therefore, } 1 - \mu\{p(\xi)=\lambda\} &= \int_{\{p(\xi)>\lambda\}} e^{(p(\xi)-\lambda)t}d\mu(\xi)
\end{aligned}$$

Since l.h.s independent of $t$ and r.h.s$\to 0$ as $t \to -\infty$, $\mu\{p(\xi) = \lambda\} = 1$.

Denote by $\nu$ the measure $(b_\lambda)_*\left(\mu|_{\{p(\xi)=\lambda\}}\right)$ on $\widetilde{M_1^\lambda}$, we have

$$e^{\lambda t}\phi_\lambda(x) = \int_{\widetilde{M_1^\lambda}} e^{\lambda t}K^\lambda(x,\xi')d\nu(\xi'),$$

for $t \leq 0$. Then, when $t = 0$,

$$\phi_\lambda(x) = \int_{\widetilde{M_1^\lambda}} K^\lambda(x,\xi')d\nu(\xi').$$

Since $\phi_\lambda$ minimal, $\nu$ must consist of a single point $\{\xi\}$, such that $\phi_\lambda = K^\lambda(x,\xi)$.

□

Now let $f(x)$ be a positive harmonic function on $M$, $f(x_0) = 1$, treat this as a positive parabolic function on $\mathcal{H}$. We have $f(x) = \int_{\widetilde{\mathcal{H}_1}} H(x,t;\xi)dm(\xi)$ for a unique Borel measure $m$ on $\widetilde{\mathcal{H}_1}$. Define $\widetilde{\mathcal{H}_1^0} = \{\xi \in \widetilde{\mathcal{H}_1} : p(\xi) = 0\}$.

PROPOSITION 4.1. $m(\widetilde{\mathcal{H}_1}\backslash\widetilde{\mathcal{H}_1^0}) = 0.$

*Proof.* $\widetilde{\mathcal{H}^+} = p^{-1}(0,\infty) \subset \widetilde{\mathcal{H}_1}$, $\widetilde{\mathcal{H}_\lambda^-} = p^{-1}(\lambda_0(M),\lambda) \subset \widetilde{\mathcal{H}_1}$, $\lambda < 0$. Then, $f(x) \geq \int_{\widetilde{\mathcal{H}_\lambda^-}} H(x,t;\xi)dm(\xi)$,

$$f(x_0) \geq \int_{\widetilde{\mathcal{H}_\lambda^-}} e^{p(\xi)t}dm(\xi).$$

Since $t \leq 0$, $\lambda_0(M) \leq \lambda < 0$, $f(x_0) \geq m(\widetilde{\mathcal{H}_\lambda^-})e^{\lambda t}$, take $t \to -\infty$, $m(\widetilde{\mathcal{H}_\lambda^-}) = 0$ for all $\lambda < 0$. Thus,

$$f(x) - \int_{\widetilde{\mathcal{H}_1^0}} H(x,t;\xi)dm(\xi) = \int_{\widetilde{\mathcal{H}^+}} H(x,t;\xi)dm(\xi),$$

evaluating at $x = x_0$, l.h.s.$=f(x_0) - m(\widetilde{\mathcal{H}_1^0}) =$ constant. Since r.h.s. is an increasing function for $t$, we have $m(\widetilde{\mathcal{H}^+}) = 0$, hence $m(\widetilde{\mathcal{H}_1}\backslash\widetilde{\mathcal{H}_1^0}) = 0$. $\square$

The following lemma was due to Cheng and Yau.

LEMMA 4.1. *Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below by a constant $K$, we have*
*(a) If $f \in C^2(M)$, $f > 0$ and $\Delta f = \lambda f$ on $M$, then $|\Delta \log f| \leq c_1 \max\{|\lambda|, |K|\}$,*
*(b) Let $f \in C^2(M)$, suppose $f$ is bounded above and does not attain its supremum on $M$. Then $\exists x_k \to \infty$ on $M$ such that $f(x_k) \to \sup f$,*
*(c) $|\nabla f|(x_k) \leq \frac{c_2}{r_k}$,*
*(d) $\Delta f(x_k) \leq \frac{c_3}{r_k}$,*
*where $r_k = d(x_k, x_0)$, $c_2$, $c_3$ depend only on the dimension of $M$ and* $\sup f$, *$c_3$ also depends on $K$.*

## 4.4 PROOF OF THEOREM 4.1

For $i = 1, 2$, let $\mathcal{H}_i = M_i \times (-\infty, 0)$, $\widetilde{\mathcal{H}^i}$ be the parabolic Martin boundary, $\widetilde{\mathcal{H}_1^i}$ be the minimal parabolic Martin boundary of $\mathcal{H}^i$, $H_i(x^i,t;y^i,s)$ and $H_i(x^i,t;\xi_i)$ be the normalized Martin kernels , $\xi_i \in \widetilde{\mathcal{H}^i}$.

Again, we let $f$ be a positive harmonic function on $M$, $f(x_0) = 1$, we have,

$$f(x) = \int_{\widetilde{\mathcal{H}_1^0}} H(x, t; \xi) dm(\xi).$$

LEMMA **4.2.** *If* $\xi \in \widetilde{\mathcal{H}_1^0}$,

$$H(x, t; \xi) = K^{\lambda_1}(x^1, \xi_1) K^{\lambda_2}(x^2, \xi_2),$$

*where* $\lambda_i \geq \lambda_0(M_i)$, $\xi_i \in \widetilde{M_1^{\lambda_i}}(M_i)$, $i = 1,\ 2$, $\lambda_1 + \lambda_2 = 0$.

*Proof.* Observing that the heat kernel of $M$ is given by

$$k_h(t, x, y) = k_{h,1}(t, x^1, y^1) k_{h,2}(t, x^2, y^2),$$

where $k_H h, i(t, x^i, y^i)$ is the heat kernel of $M_i$, $i = 1,\ 2$. Hence we have

$$H(x, t; y, s) = H_1(x^1, t; y^1, s) H_2(x^2, t; y^2, s).$$

Now let $\{y_i, s_i\}$ be a fundamental sequence converging to $\xi \in \widetilde{\mathcal{H}}$, $y_i = (y_i^1, y_i^2)$. Passing to subsequence if necessary, we have $\{y_i^1, s_i\} \to \xi_1 \in \widetilde{\mathcal{H}^1}$, $\{y^2, s_i\} \to \xi_2 \in \widetilde{\mathcal{H}^2}$, i.e.,

$$H(x, t; \xi) = H_1(x^1, t; \xi_1) H_2(x^2, t; \xi_2)$$

In can be easily seen that if $\xi \in \widetilde{\mathcal{H}_1}$, then $\xi_i \in \widetilde{M_1^{\lambda_i}}(M_i)$, $i = 1, 2$. Thus, Theorem 4.3 implies,

$$H(x, t; \xi) = e^{\lambda_1 t} K^{\lambda_1}(x^1, \xi_1) e^{\lambda_2 t} K^{\lambda_2}(x^2, \xi_2).$$

$\lambda_1 + \lambda_2 = 0$, thus, for $\xi \in \widetilde{\mathcal{H}_1^0}$,

$$H(x, t; \xi) = K^{\lambda_1}(x^1, \xi_1) K^{\lambda_2}(x^2, \xi_2).$$

□

If $f$ is minimal positive harmonic, the support of $m$ consists of a unique point, therefore,

$$f(x) = H(x,t;\xi') = K^{\lambda_1}(x^1,\xi_1')K^{\lambda_2}(x^2,\xi_2').$$

Now, let $g$ be a positive harmonic function on $M$, $g(x_0) = 1$ such that $g \leq K^{\lambda_1}(x^1)K^{\lambda_2}(x^2)$. We want to show that in fact $g = K^{\lambda_1}(x^1)K^{\lambda_2}(x^2)$. Let $m$ be the measure on $\widetilde{\mathcal{H}_1^0}$ such that

$$g(x) = \int_{\widetilde{\mathcal{H}_1^0}} H(x,t;\xi)dm(\xi).$$

For $\lambda > \lambda_0(M_1)$, define the following:

$$\begin{aligned} E_\lambda^1 &= \{v \in P(\mathcal{H}) : v \geq 0 \text{ and } v(x^1,x_0^2,-1) \text{ is } \lambda-\text{subharmonic on } M_1\}, \\ F_\lambda^1 &= \{v \in P(\mathcal{H}) : v \geq 0 \text{ and } \Delta_i v(x^1,x_0^2,-1) = \lambda v(x^1,x_0^2,1)\}, \end{aligned}$$

and similarly for $E_\lambda^2$ and $F_\lambda^2$. Denote $F_{\lambda_1,\lambda_2} = F_{\lambda_1}^1 \cap F_{\lambda_2}^2$. By theorem 4.3, we can write $\widetilde{\mathcal{H}_1^0}$ as a disjoint union:

$$\widetilde{\mathcal{H}_1^0} = \left(\cup_{\lambda_n \downarrow \lambda_1}\left(E_{\lambda_n}^1 \cap \widetilde{\mathcal{H}_1^0}\right)\right) \bigcup \left(F_{\lambda_1,\lambda_2} \cap \widetilde{\mathcal{H}_1^0}\right) \bigcup \left(\cup_{\mu_n \downarrow \lambda_1}\left(E_{\mu_n}^2 \cap \widetilde{\mathcal{H}_1^0}\right)\right)$$

CLAIM **4.1.** *For any $\lambda > \lambda_1$, $m(E_\lambda^1 \cap \widetilde{\mathcal{H}_1^0}) = 0$.*

*Proof.* Suppose not, then, let

$$w(x,t) = \int_{E_\lambda^1 \cap \widetilde{\mathcal{H}_1^0}} H(x,t;\xi)dm(\xi).$$

We have $K^{\lambda_1}K^{\lambda_2} \geq w > 0$ and $h \in E_\lambda^1$. Define $v(x^1) = w(x^1,x_0^2,-1)$, $v \in C^\infty(M_1)$, $0 < v \leq K^{\lambda_1}$ and $\Delta_1 v \geq \lambda v$. Now let $h(x^1) = \dfrac{v}{K^{\lambda_1}}$. Then, $h$ satisfies $0 < h \leq 1$ and

$$\begin{aligned} \Delta_1 h &= \frac{\Delta_1 v}{K^{\lambda_1}} + 2 < \nabla_1 v, \nabla_1 \log K^{\lambda_1} > -h\frac{\Delta_1 K^{\lambda_1}}{K^{\lambda_1}} \\ &\geq (\lambda - \lambda_1)h - 2|\nabla_1 h||\nabla_1 \log K^{\lambda_1}| \\ &\geq (\lambda - \lambda_1)h - c|\nabla_1 h|, \end{aligned}$$

by part (a) of Lemma 4.1. This inequality implies that $h$ does not have an positive interior maximum. Therefore, by part (b) of Lemma 4.1, there exists $\{x_k\} \to \infty$ such that $h(x_k) \to \sup h$ together with part (c) and (d) of Lemma 4.1, we have $|\nabla_1 h|(x_k) \le c_2/r_k$ and $\Delta_1 h(x_k) \le c_3/r_k$, hence $h(x_k) \to 0$. This implies $h \equiv 0$ and is a contradiction. Therefore, we have $m(E^1_\lambda \cap \widetilde{\mathcal{H}^0_1}) = 0$ and similarly $m(E^2_{\lambda'} \cap \widetilde{\mathcal{H}^0_1}) = 0$ for $\lambda' > \lambda_2$. □

Now, we have

$$g(x) = \int_{F_{\lambda_1,\lambda_2} \cap \widetilde{\mathcal{H}^0_1}} H(x,t;\xi) dm(\xi),$$

$\lambda_1 + \lambda_2 = 0$. Therefore, $\Delta_2 g = \lambda_2 g$. Since $g(x^1_0, x^2) \le K^{\lambda_1}(x^1_0) K^{\lambda_2}(x^2) = K^{\lambda_2}(x^2)$, $K^{\lambda_2}$ minimal implies $g(x^1_0, x^2) = K^{\lambda_2}(x^2)$. On the other hand, $\Delta_1 g = \lambda_1 g$, $g(x^1, x^2) \le K^{\lambda_1}(x^1) g(x^1_0, x^2)$ and $K^{\lambda_1}$ minimal implies $g(x^1, x^2) = K^{\lambda_1}(x^1) g(x^1_0, x^2)$, i.e., $g(x^1, x^2) = K^{\lambda_1}(x^1) K^{\lambda_2}(x^2)$. Hence, $K^{\lambda_1}(x^1) K^{\lambda_2}(x^2)$ is minimal, and completes the proof.

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